School of Engineering
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EN2210: Continuum Mechanics

Homework 6: Constitutive Equations, Fluid Mechanics
Solutions

1. Show that a fluid with constitutive equation of the form

$$\sigma_{ij} = -\pi_{eq}(\rho,\theta)\delta_{ij} + \tau_{ij}^0 + 2\mu D_{ij}$$

with τ_{ij}^0 a nonzero constant, violates the second law of thermodynamics.

The second law requires

$$\sigma_{ij} D_{ij} - \frac{1}{\theta} q_i \frac{\partial\theta}{\partial y_i} - \rho\left(\frac{\partial\psi}{\partial\rho}\dot{\rho} + \frac{\partial\psi}{\partial\theta}\dot{\theta} + \frac{\partial\psi}{\partial D_{ij}}\dot{D}_{ij} + s\frac{\partial\theta}{\partial t}\right) \ge 0$$

For all processes. We can consider a process with constant density and temperature, which gives

$$\sigma_{ij} D_{ij} \ge 0$$

$$\sigma_{ij} D_{ij} = -\pi_{eq}(\rho,\theta) D_{kk} + \tau_{ij}^0 D_{ij} + 2\mu D_{ij} D_{ij}$$

But $D_{kk} = 0$ for a constant density process. We can choose $D_{ij} = -\lambda(\tau_{ij}^0 - \tau_{kk}^0\delta_{ij}/3)$ ($\lambda > 0$) (we need to make **D** volume preserving) in which case

$$\begin{aligned}\sigma_{ij} D_{ij} &= -\tau_{ij}^0 \lambda(\tau_{ij}^0 - \tau_{kk}^0\delta_{ij}/3) + 2\mu\lambda^2(\tau_{ij}^0 - \tau_{kk}^0\delta_{ij}/3)(\tau_{ij}^0 - \tau_{nn}^0\delta_{ij}/3) \\ &= -\lambda\tau_{ij}^0\tau_{ij}^0 + \lambda\tau_{kk}^0\tau_{nn}^0/3 + 2\mu\lambda^2\tau_{ij}^0\tau_{ij}^0 - 2\mu\lambda^2\tau_{kk}^0\tau_{nn}^0/3 \\ &= -\left(\lambda - 2\mu\lambda^2\right)\left(\tau_{ij}^0\tau_{ij}^0 - \tau_{kk}^0\tau_{nn}^0/3\right)\end{aligned}$$

If $\tau_{ij}^0\tau_{ij}^0 - \tau_{kk}^0\tau_{nn}^0/3 > 0$ this is negative for any $\lambda < \dfrac{1}{2\mu}$. If $\tau_{ij}^0\tau_{ij}^0 - \tau_{kk}^0\tau_{nn}^0/3 < 0$ it is negative for any $\lambda > 1/2\mu$. Hence there is always some **D** for which the second law is violated.

[3 POINTS]

2. Suppose that the internal energy of a continuum is expressed as a function of density and entropy, as $\varepsilon(\rho,s)$. Show that the dissipation inequality requires that

$$\sigma_{ij}D_{ij}-\frac{1}{\theta}q_i\frac{\partial\theta}{\partial y_i}-\rho\left(\frac{\partial\varepsilon}{\partial t}-\theta\frac{\partial s}{\partial t}\right)\ge 0$$

The standard dissipation in equality is

$$\sigma_{ij}D_{ij}-\frac{1}{\theta}q_i\frac{\partial\theta}{\partial y_i}-\rho\left(\frac{\partial\psi}{\partial t}+s\frac{\partial\theta}{\partial t}\right)\ge 0$$

Substituting $\psi=\varepsilon-\theta s$ gives the required result.

[3 POINTS]

3. Consider an inviscid van der Waals fluid with specific heat capacity $c_v(\theta)$ an arbitrary function of temperature (but independent of density), and pressure related to temperature and density by

$$\hat{\pi}_{eq}=\frac{\rho R\theta}{1-b\rho}-a\rho^2$$

3.1 Show that the dissipation inequality (use problem 2) requires that

$$\frac{d\varepsilon}{dt}=\hat{\pi}_{eq}\frac{1}{\rho^2}\frac{d\rho}{dt}+\theta\frac{ds}{dt}$$

For this case the dissipation inequality reduces to

$$=-\pi_{eq}D_{kk}-\frac{1}{\theta}q_i\frac{\partial\theta}{\partial y_i}-\rho\left(\frac{\partial\varepsilon}{\partial t}-\theta\frac{\partial s}{\partial t}\right)\ge 0$$

$$\Rightarrow \pi_{eq}\frac{\dot{\rho}}{\rho}-\frac{1}{\theta}q_i\frac{\partial\theta}{\partial y_i}-\rho\left(\frac{\partial\varepsilon}{\partial t}-\theta\frac{\partial s}{\partial t}\right)\ge 0$$

$$\Rightarrow \pi_{eq}\frac{\dot{\rho}}{\rho}-\rho\frac{\partial\varepsilon}{\partial\rho}\dot{\rho}-\rho\frac{\partial\varepsilon}{\partial s}\dot{s}+\theta\rho\frac{\partial s}{\partial t}-\frac{1}{\theta}q_i\frac{\partial\theta}{\partial y_i}\ge 0$$

This must hold for all $\dot{s},\dot{\rho}$ which implies that

$$\hat{\pi}_{eq}=\rho^2\frac{\partial\varepsilon}{\partial\rho}\qquad \theta=\frac{\partial\varepsilon}{\partial s}$$

This now shows that

$$\pi_{eq}\frac{\dot{\rho}}{\rho}-\rho\frac{\partial\varepsilon}{\partial\rho}\dot{\rho}-\rho\frac{\partial\varepsilon}{\partial s}\dot{s}+\theta\rho\frac{\partial s}{\partial t}=0$$

And hence

$$\frac{d\varepsilon}{dt}=\hat{\pi}_{eq}\frac{1}{\rho^2}\frac{d\rho}{dt}+\theta\frac{ds}{dt}$$

[3 Points]

3.2 Hence conclude that

$$\frac{d(\varepsilon+a\rho)}{dt}=\theta\frac{d}{dt}\left(R\log\frac{\rho}{1-b\rho}+s\right)$$

for all $\dot{\rho}, \dot{\theta}$

Substitute for $\hat{\pi}_{eq}$

$$\frac{d\varepsilon}{dt} = \left(\frac{\rho R\theta}{1-b\rho} - a\rho^2\right)\frac{1}{\rho^2}\frac{d\rho}{dt} + \theta\frac{ds}{dt}$$

Rearranging this result gives the required expression.

[3 POINTS]

3.3 Hence show that

$$s = \int \frac{c_v(\theta)}{\theta} d\theta - R\log\frac{\rho}{1-b\rho} + const$$

$$\varepsilon = \theta\int \frac{c_v(\theta)}{\theta} d\theta - a\rho + const$$

Let $\left(R\log\frac{\rho}{1-b\rho} + s\right) = \phi$, then the preceding problem shows that

$$\frac{\partial\varepsilon}{\partial\rho}\dot{\rho} + \frac{\partial\varepsilon}{\partial\theta}\dot{\theta} = \frac{\partial\varepsilon}{\partial\rho}\dot{\rho} + c_v\dot{\theta} = \theta\left(\frac{\partial\phi}{\partial\rho}\dot{\rho} + \frac{\partial\phi}{\partial\theta}\dot{\theta}\right)$$

This must hold for all $\dot{\theta}, \dot{\rho}$ so

$$\frac{c_v}{\theta}\dot{\theta} = \frac{\partial\phi}{\partial\theta}\dot{\theta}$$

Integrating this expression gives

$$\phi = \int \frac{c_v(\theta)}{\theta} d\theta + const$$

which gives the first result.

For the second one note that

$$\frac{\partial\varepsilon}{\partial\rho} + a = \theta\frac{\partial\phi}{\partial\rho} \Rightarrow \varepsilon = \theta\phi(\theta) - a\rho + g(\theta)$$

But

$$\frac{\partial\varepsilon}{\partial\theta} = c_v \Rightarrow \varepsilon = \theta\phi(\theta) + g(\theta) = \theta\int \frac{c_v}{\theta} d\theta$$

[3 POINTS]

4. The deformation of a viscoelastic material is modeled by representing the deformation gradient **F** of a material element as a sequence of an irreversible deformation $\mathbf{F}^p$, followed by a reversible (elastic) deformation $\mathbf{F}^e$, so that $\mathbf{F}=\mathbf{F}^e\mathbf{F}^p$. The Helmholtz free energy $\psi(\mathbf{F}^e,\theta)$ of the material is assumed to be a function of $\mathbf{F}^e$ and temperature θ only.

4.1 Show that the velocity gradient **L** can be decomposed into elastic and plastic parts as

$$\mathbf{L}=\mathbf{L}^e+\mathbf{L}^p \qquad \mathbf{L}^e=\frac{d\mathbf{F}^e}{dt}\mathbf{F}^{e-1} \qquad \mathbf{L}^p=\mathbf{F}^e\frac{d\mathbf{F}^p}{dt}\mathbf{F}^{p-1}\mathbf{F}^{e-1}$$

We have that $\mathbf{L}=\frac{d\mathbf{F}}{dt}\mathbf{F}^{-1}=\frac{d(\mathbf{F}^e\mathbf{F}^p)}{dt}\mathbf{F}^{p-1}\mathbf{F}^{e-1}$, and expanding the time derivative using the product rule and simplifying gives the required solution.

[3 POINTS]

4.2 Show that the dissipation inequality

$$\sigma_{ij}D_{ij}-\frac{1}{\theta}q_i\frac{\partial\theta}{\partial y_i}-\rho\left(\frac{\partial\psi}{\partial t}+s\frac{\partial\theta}{\partial t}\right)\geq 0$$

requires that the Cauchy stress is related to the free energy by

$$JF_{kj}^{e-1}\sigma_{ji}=\rho_0\frac{\partial\psi}{\partial F_{ik}^e}$$

(where ρ_0 is the mass per unit reference volume) and that the plastic part of the velocity gradient must satisfy

$$\sigma_{ij}L_{ij}^p\geq 0$$

Noting that $\sigma_{ij}D_{ij}=\sigma_{ij}L_{ij}$ from the symmetry of the Cauchy stress, and taking the time derivative of the free energy gives

$$\sigma_{ij}\frac{dF_{jk}^e}{dt}F_{ki}^{e-1}+\sigma_{ij}L_{ij}^p-\frac{1}{\theta}q_i\frac{\partial\theta}{\partial y_i}-\frac{\rho_0}{J}\left(\frac{\partial\psi}{\partial F_{ij}^e}\frac{dF_{ij}^e}{dt}+s\frac{\partial\theta}{\partial t}\right)\geq 0$$

This must hold for all $\dot{F}_{ij}^e,\dot{F}_{ij}^p$, which shows that

$$\sigma_{ij}\frac{dF_{jk}^e}{dt}F_{ki}^{e-1}-\frac{\rho_0}{J}\frac{\partial\psi}{\partial F_{ij}^e}\frac{dF_{ij}^e}{dt}\geq 0\Rightarrow\left(F_{ki}^{e-1}\sigma_{ij}-\frac{\rho_0}{J}\frac{\partial\psi}{\partial F_{jk}^e}\right)\frac{dF_{jk}^e}{dt}\Rightarrow JF_{ki}^{e-1}\sigma_{ij}=\rho_0\frac{\partial\psi}{\partial F_{jk}^e}$$

and $\sigma_{ij}L_{ij}^p\geq 0$ follows directly

[3 POINTS]

4.3 Assume that $\mathbf{F}^e$ and $\mathbf{F}^p$ transform under a change of observer according to $\mathbf{F}^{e*}=\mathbf{Q}\mathbf{F}^e \qquad \mathbf{F}^{p*}=\mathbf{F}^p$. Verify that the transformation is consistent with the transformation of deformation gradient **F** under an observer change, and determine expressions for $\mathbf{L}^{e*},\mathbf{L}^{p*}$ in terms of **Q** and $\mathbf{\Omega}=\dot{\mathbf{Q}}\mathbf{Q}^T$.

The deformation gradient should transform as $\mathbf{F}^*=\mathbf{Q}\mathbf{F}$. For the transformations given we have

$$\mathbf{F}^* = \mathbf{F}^{e*}\mathbf{F}^{p*} = \mathbf{Q}\mathbf{F}^e\mathbf{F}^p = \mathbf{Q}\mathbf{F}$$

We also have that $\mathbf{L}^{e*} = \dot{\mathbf{F}}^{e*}\mathbf{F}^{e*-1} = \left(\dot{\mathbf{Q}}\mathbf{F}^e + \mathbf{Q}\dot{\mathbf{F}}^e\right)\mathbf{F}^{e-1}\mathbf{Q}^T = \mathbf{\Omega} + \mathbf{Q}\mathbf{L}^e\mathbf{Q}^T \quad \mathbf{L}^{p*} = \mathbf{Q}\mathbf{L}^p\mathbf{Q}^T$

[3 POINTS]

4.4 Consider a constitutive relation in which the plastic velocity gradient is given by

$$L_{ij}^p = \eta\left(\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}\right)$$

Show that if $\det(\mathbf{F}^p) = 1$ at time *t=0*, then $\det(\mathbf{F}^p) = 1$ for all *t>0*. (Hint: consider L_{kk}^p)

In the usual way,

$$\frac{dJ_p}{dt} = J_p \frac{dF_{kn}^p}{dt} F_{nk}^{p-1}$$

Note also that

$$L_{kk}^p = F_{ik}^e \frac{dF_{kn}^p}{dt} F_{nm}^{p-1} F_{mi}^{e-1} = \frac{dF_{kn}^p}{dt} F_{nk}^{p-1}$$

Finally the constitutive equation shows that $L_{kk}^p = 0$.

[3 POINTS]

4.5 Show that the constitutive relation in 3.4 satisfies both frame indifference and the dissipation inequality (assume $\eta > 0$).

Note that $\sigma_{ij}L_{ij}^p = \eta\sigma_{ij}\left(\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}\right) = \eta\left(\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}\right)\left(\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}\right)$. This is a perfect square.

The constitutive equation satisfies

$$\mathbf{Q}\mathbf{L}^p\mathbf{Q}^T = \eta\left(\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T - \frac{1}{3}tr(\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T)\mathbf{I}\right) \Rightarrow \mathbf{L}^{p*} = \eta\left(\boldsymbol{\sigma}^* - \frac{1}{3}tr(\boldsymbol{\sigma}^*)\mathbf{I}\right)$$

[3 POINTS]