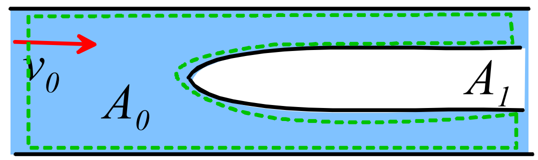
School of Engineering
Brown University

EN2210: Continuum Mechanics

Homework 7: Fluid Mechanics
Solutions

1. An ideal fluid with mass density ρ flows with velocity v_0 through a cylindrical tube with cross-sectional area A_0 . A body extending downstream to infinity with cross-sectional area A_1 . Find an expression for the force acting on the body in the tube



Consider the control volume shown in the figure. Let F denote the force acting on the area $A_0 - A_1$ (positive force acts to the right)

Mass conservation gives $v_0 A_0 = v_1 (A_0 - A_1)$

Linear momentum gives
$F + p_0 A_0 - p_1 (A_0 - A_1) = -\rho v_0 v_0 A_0 + \rho v_1 v_1 (A_0 - A_1)$
(the left hand side is the integral of the traction on the control vol; the right hand side is the rate of change of linear momentum.)

Bernoulli gives
$p_0 + \frac{1}{2}\rho v_0^2 = p_1 + \frac{1}{2}\rho v_1^2$

Solving these equations gives

$$F = -A_1 p_0 + \rho v_0^2 \frac{A_1^2}{2(A_0 - A_1)}$$

The force on the body is equal and opposite….

[3 POINTS]

2. An incompressible Newtonian viscous fluid with viscosity η and density ρ occupies the region $y_2 < 0$. At time *t=0* the fluid has velocity distribution $v_1 = u_0 \exp(-y_2^2 / b^2)$ with all other velocity components zero. The goal of this problem is to calculate the velocity in the fluid for *t=0*. Gravity and pressure variations in the fluid may be neglected. By considering a velocity field of the form $v_1 = u_0 f(t) \exp(-y_2^2 f(t)^2 / b^2)$, calculate the variation of velocity in the fluid with position and time.

The Navier-Stokes equation reduces to

$$+\frac{\eta}{\rho}\left\{\frac{\partial^2 v_1}{\partial y_2^2}\right\}=\frac{\partial v_1}{\partial t}$$

$$\Rightarrow \frac{\eta}{\rho}u_0\frac{\partial}{\partial y_2}\left\{-2f(t)^3\frac{y_2}{b^2}\exp(-y_2^2 f(t)^2/b^2)\right\}=u_0 f'(t)\left(1-2\frac{y_2^2 f(t)^2}{b^2}\right)\exp(-y_2^2 f(t)^2/b^2)$$

$$\Rightarrow \frac{\eta}{\rho b^2}f(t)^3 u_0\left\{-2\exp(-y_2^2 f(t)^2/b^2)+4f(t)^2\frac{y_2^2}{b^2}\exp(-y_2^2 f(t)^2/b^2)\right\}$$

$$=u_0 f'(t)\left(1-2\frac{y_2^2 f(t)^2}{b^2}\right)\exp(-y_2^2 f(t)^2/b^2)$$

$$\Rightarrow -\frac{2\eta}{\rho b^2}f(t)^3=f'(t)\Rightarrow \frac{1}{f(t)^2}=\frac{4\eta t}{\rho b^2}+C$$

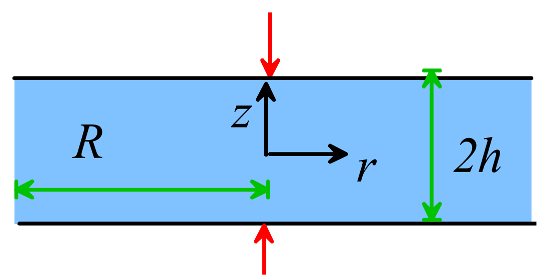
The initial condition gives *C=1*, which gives

$$v_1=\frac{u_0}{\sqrt{1+4\eta t/\rho b^2}}\exp(-y_2^2/(b^2+4\eta t/\rho))$$

[3 POINTS]

3. Two parallel circular plates with radius *R* are separated by an incompressible Newtonian fluid with viscosity η. The plates are a distance $2h$ apart and approach one another slowly with relative speed $2U$. The goal of this problem is to find an approximate solution for the velocity distribution in the fluid between the plates. Assume Stokes flow and neglect the acceleration, and assume a velocity field of the form

$v_r = rf(z) \qquad v_z = g(z)$



3.1 Use the incompressibility condition to determine $f(z)$ in terms of $g(z)$

Incompressibility requires $\nabla\cdot\mathbf{v}=0$, which gives

$$\frac{\partial v_r}{\partial r}+\frac{v_r}{r}+\frac{\partial v_z}{\partial z}=2f(z)+\frac{dg}{dz}=0\Rightarrow f(z)=-\frac{1}{2}\frac{dg}{dz}$$

[2 POINTS]

3.2 Hence, find an expression for the velocity gradient in terms of $g(z)$ and *r* (use cylindrical polar coordinates)

$$\mathbf{v} = -\frac{1}{2} r \frac{dg}{dz} \mathbf{e}_r + g \mathbf{e}_z$$

$$\nabla \mathbf{v} = \frac{\partial}{\partial r}\left(-\frac{1}{2} r \frac{dg}{dz}\right) \mathbf{e}_r \otimes \mathbf{e}_r - \frac{1}{2}\frac{dg}{dz} \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \frac{\partial}{\partial z}\left(-\frac{1}{2} r \frac{dg}{dz}\right) \mathbf{e}_r \otimes \mathbf{e}_z + \frac{dg}{dz} \mathbf{e}_z \otimes \mathbf{e}_z$$

[2 POINTS]

3.3 Write down *two* boundary conditions for $g(z)$ at $z = \pm h$

The conditions $v_r = v_z = 0$ give $g(\pm h) = \mp U \qquad \frac{dg}{dz} = 0$

[2 POINTS]

3.4 Show that for this problem the principle of virtual work reduces to

$$\int_0^R \int_{-h}^h 2\eta \mathbf{D} : \nabla \delta \tilde{\mathbf{v}} \; dz \; 2\pi r dr = 0 \qquad \forall \; \delta \tilde{\mathbf{v}} \; : \nabla \cdot \delta \tilde{\mathbf{v}} = 0$$

$$\int_0^R \int_{-h}^h \boldsymbol{\sigma} : \delta \mathbf{D} \; dz \; 2\pi r dr = \int_0^R \int_{-h}^h 2\eta \mathbf{D} : \delta \mathbf{D} \; dz \; 2\pi r dr = 0$$

$$\Rightarrow \int_0^R \int_{-h}^h 2\eta \mathbf{D} : (\delta \mathbf{L} - \delta \mathbf{W}) dz \; 2\pi r dr \Rightarrow \int_0^R \int_{-h}^h 2\eta \mathbf{D} : \delta \mathbf{L} \, dz \; 2\pi r dr = 0$$

[2 POINTS]

3.5 Hence, show that

$$\int_0^R dr \int_{-h}^h 2\pi r \left[\left(\frac{3}{2}\frac{dg}{dz}\right)\left(\frac{d\delta g}{dz}\right) + \frac{r^2}{8}\left(\frac{d^2 g}{dz^2}\right)\left(\frac{d^2 \delta g}{dz^2}\right) \right] dz = 0$$

$$\int_0^R \int_{-h}^{h} 2\eta \mathbf{D} : \delta\mathbf{D}\ dz\ 2\pi r dr = 0$$

$$\Rightarrow \int_0^R \int_{-h}^{h} 2\eta \begin{bmatrix} L_{rr} & 0 & 0 \\ 0 & L_{\theta\theta} & L_{rz}/2 \\ 0 & L_{rz}/2 & L_{zz} \end{bmatrix} : \begin{bmatrix} \delta L_{rr} & 0 & 0 \\ 0 & \delta L_{\theta\theta} & \delta L_{rz}/2 \\ 0 & \delta L_{rz}/2 & \delta L_{zz} \end{bmatrix} dz\ 2\pi r dr = 0$$

$$\Rightarrow \int_0^R \int_{-h}^{h} 2\eta \left[\left(\frac{1}{2}\frac{dg}{dz}\right)\left(\frac{1}{2}\frac{d\delta g}{dz}\right) + \frac{1}{2}\frac{dg}{dz}\frac{1}{2}\frac{d\delta g}{dz} + \frac{2}{4}\frac{\partial}{\partial z}\left(\frac{1}{2}r\frac{dg}{dz}\right)\frac{\partial}{\partial z}\left(\frac{1}{2}r\frac{d\delta g}{dz}\right) + \frac{dg}{dz}\frac{d\delta g}{dz} \right] 2\pi r dr$$

$$\int_0^R \int_{-h}^{h} 2\eta \left[\left(\frac{3}{2}\frac{dg}{dz}\right)\left(\frac{d\delta g}{dz}\right) + \frac{1}{2}\frac{\partial}{\partial z}\left(\frac{1}{2}r\frac{dg}{dz}\right)\frac{\partial}{\partial z}\left(\frac{1}{2}r\frac{d\delta g}{dz}\right) \right] 2\pi r dr$$

[2 POINTS]

3.6 Deduce that

$$\frac{d^2 g}{dz^2} - \frac{R^2}{24}\frac{d^4 g}{dz^4} = 0$$

and hence determine *g(z)*.

Integrate the first term in the integral by parts once, the second term by parts twice, and use the condition that $\delta g = \frac{d\delta g}{dz} = 0$

$$\int \int_{-h}^{h} \left[\frac{d}{dz}\left(\frac{3}{2}\frac{dg}{dz}\right) - \frac{1}{8}r^2 \frac{d^4 g}{dz^4} \right] \delta g 2\pi r dr$$

$$\int_{-h}^{h} \left[\frac{R^2}{2}\frac{d}{dz}\left(\frac{3}{2}\frac{dg}{dz}\right) - \frac{1}{8}\frac{R^4}{4}\frac{d^4 g}{dz^4} \right] \delta g dz 2\pi$$

$$\frac{3}{4}\frac{d^2 g}{dz^2} - \frac{R^2}{32}\frac{d^4 g}{dz^4} = 0$$

This equation can be solved to give

$$g(z) = \frac{2\sqrt{6}Uz\left(e^{2\sqrt{6}h/R} + e^{-2\sqrt{6}h/R}\right) + RU\left(e^{-2\sqrt{6}z/R} - e^{2\sqrt{6}z/R}\right)}{2\sqrt{6}h\left(e^{-2\sqrt{6}h/R} + e^{2\sqrt{6}h/R}\right) + R\left(e^{-2\sqrt{6}h/R} - e^{2\sqrt{6}h/R}\right)}$$

[3 POINTS]

3.7 Show that for $R >> h$ the solution can be approximated further by $g(z) = U(z^3/h^3 - 3z/h)/2$

The Taylor expansion of *g(z)* gives the required result.

[2 POINTS]

3.8 For the approximation in 3.6, calculate the pressure distribution in the fluid, and deduce an expression for the force required to squeeze the plates together (assume the integral of the pressure through the film thickness at *r*=*R* vanishes). You can assume that the Stokes equation for a cylindrically symmetric flow is

$$-\frac{\partial p}{\partial r}+\eta\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right)-\frac{v_r}{r^2}+\frac{\partial^2 v_r}{\partial z^2}\right)=0$$

$$-\frac{\partial p}{\partial z}+\eta\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{\partial^2 v_z}{\partial z^2}\right)=0$$

These equations can be integrated to get

$$p=\frac{3U\eta}{2h}\left[\frac{z^2}{h^2}-\frac{1}{3}-\frac{(r^2-R^2)}{2h^2}\right]\approx\frac{3U\eta}{2h}\frac{(R^2-r^2)}{2h^2}$$

[2 POINTS]

The resultant force is

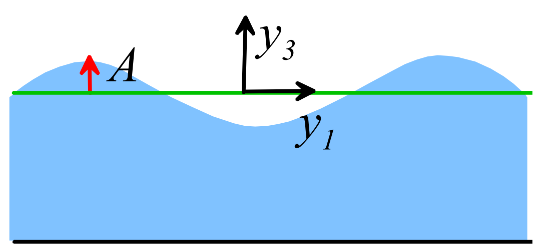
$$\frac{\pi R^2 U\eta}{8h^3}\left(3R^2+8h^2\right)\approx\frac{3\pi R^4 U\eta}{8h^3}$$

[1 POINT]

4. The potential

$$\phi=\frac{\omega}{k}A\frac{\cosh(k(y_3+h))}{\sinh(kh)}\sin(ky_1-\omega t)$$

describes small amplitude surface waves in an ideal fluid with depth h .



4.1 Calculate the velocity distribution in the fluid

$$v_1=A\omega\frac{\cosh(k(y_3+h))}{\sinh(kh)}\cos(ky_1-\omega t)$$

$$v_3=A\omega\frac{\sinh(k(y_3+h))}{\sinh(kh)}\sin(kx_1-\omega t)$$

[2 POINTS]

4.2 Show that the velocity at the surface of the fluid is consistent with a surface wave with a vertical displacement $w_2=A\cos(kx_1-\omega t)$. What are the trajectories of material particles in the fluid?

On $y_3 = 0$ we have $v_2 = A\omega \sin(ky_1 - \omega t) = \frac{\partial w_2}{\partial t}$

Material particles describe circular orbits

[2 POINTS]

4.3 Find an expression for the pressure in the fluid (neglect the term of order a^2)

$$p + \rho \frac{\partial \phi}{\partial t} + \rho g y_3 = 0 \Rightarrow p = -\rho \left(g y_3 - \frac{\omega^2}{k} a \frac{\cosh(k(y_3 + h))}{\sinh(kh)} \cos(ky_1 - \omega t) \right)$$

[2 POINTS]

4.4 For small amplitude waves the boundary condition on the fluid surface can be expressed as

$$p(y_3 = 0) + \left. \frac{\partial p}{\partial y_3} \right|_{x_3=0} A\cos(ky_1 - \omega t) \approx 0$$

Find a condition relating k and ω (the dispersion relation) necessary to satisfy this condition (neglect the term of order A^2)

Substituting into the boundary condition we get

$$A\cos(ky_1 - \omega t)\left(\frac{\omega^2}{k\tanh(kh)} - g \right) = 0 \Rightarrow \omega^2 = gk\tanh(kh)$$

[2 POINTS]

Water waves are a rich source of math problems: This paper is an example

5. By considering a flow potential of the form $\Omega = A\sin\omega t \sin ky_1$ calculate the natural frequencies of a 1-D air column with sound speed c_s , for the following boundary conditions:
(i) The end at $y_1 = 0$ is open (constant pressure); while the end at $y_1 = L$ is closed (zero velocity).
(ii) Both ends are open.

The governing equation is

$$\frac{\partial^2 \Omega}{\partial t^2} - c_s^2 \frac{\partial^2 \Omega}{\partial y_1^2} = 0 \Rightarrow \omega^2 - k^2 c_s^2 = 0$$

The boundary conditions give

$$v_1 = \frac{\partial \Omega}{\partial y_1} = 0 \qquad \text{closed end}$$

$$p = -\rho \frac{\partial \Omega}{\partial t} \quad \text{or} \quad \frac{\partial p}{\partial t} = -\rho c_s^2 \frac{\partial v_1}{\partial y_1} = 0 \Rightarrow \frac{\partial^2 \Omega}{\partial y_1^2} = 0 \quad y_1 = 0 \qquad \text{open end}$$

For the closed end the second boundary condition is satisfied by the chosen form for the solution; the first is satisfied by choosing $\cos kL = 0 \Rightarrow k = \frac{(\pi + 2n)}{2L} \qquad n = 0,1,2...$

The frequencies are thus $\omega_n = \frac{(\pi + 2n)}{2L} c_s \qquad n = 0,1,2...$

Similarly if both ends are open then

$$\omega_n = \frac{n\pi}{L} c_s \qquad n = 1,2...$$

[2 POINTS]

(iii) Estimate the length of a transverse flute whose lowest frequency is 246.9 Hz (both ends of a flute are open. Sound speed in air is 343.2 m/s)

$$2\pi f = c_s \frac{\pi}{L} \Rightarrow L = \frac{c_s}{2f} \approx 70cm$$

[1 POINT]