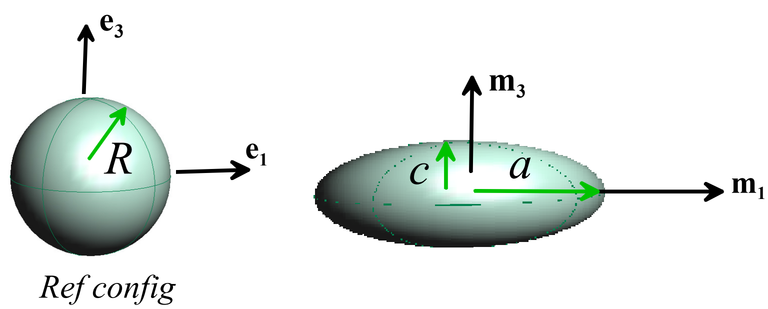
School of Engineering
Brown University

EN2210: Continuum Mechanics

Homework 8: Elasticity
Due Wednesday Nov 30 2016



1. An incompressible neo-hookean sphere with radius R, modulus μ and mass density ρ is deformed into an ellipsoid with semi-axes $a = \lambda_1 R \quad b = \lambda_1 R \quad c = \lambda_2 R$.

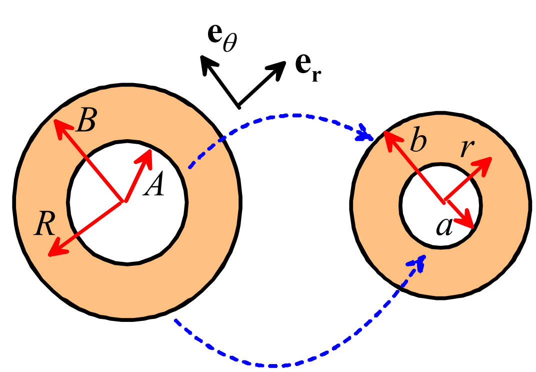
1.1 Use the incompressibility condition to express λ_2 in terms of λ_1

1.2 Find an expression for the total elastic strain energy in the solid, in terms of λ_1

1.3 Find an expression for the total kinetic energy of the solid, in terms of λ_1 and its time derivatives. Assume the center is stationary.

1.4 Hence, estimate the natural frequency of small amplitude oscillations of this (approximate) vibration mode

2. A rubber tube with internal radius A and external radius B is turned inside out, so that the surfaces at $R=A$, $R=B$ now lie at deformed radius $r=b$, $r=a$, respectively. These surfaces are free of traction. Assume that plane cross-sections of the tube remain plane, ***and that the length of the cylinder does not change***. Note that a cross section at distance Z along the axis of the undeformed tube moves to a new position $z=-Z$ after deformation. The tube may be idealized as an incompressible, neo-Hookean material with stress-strain relation



$$\boldsymbol{\sigma} = \mu \mathbf{B} + p(r)\mathbf{I}$$

2.1 By considering the volumes of material in the annular regions between r and a, and between R and B, find an expression for the radial position r of a material particle that starts at radius R in the tube before deformation.

2.2 Show that the deformation gradient $\mathbf{F}$ is given by

$$\mathbf{F} = \begin{bmatrix} -\dfrac{R}{r} & 0 & 0 \\ 0 & \dfrac{r}{R} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and hence write down an expression for the stress in the tube as a function of r, R and p.

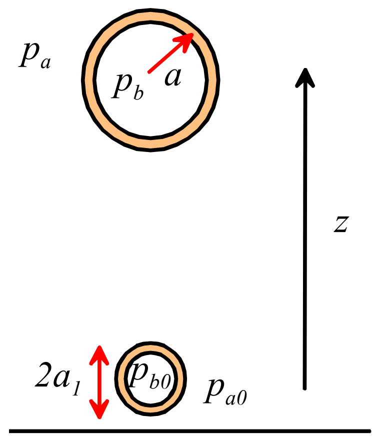
2.3 By considering a virtual velocity $\delta \mathbf{v} = v(r)\mathbf{e}_r$ where $v(r)$ is a continuously differentiable function, show that the principle of virtual work requires that

$$\int_a^b \left[\frac{d}{dr}\left\{ r\left(\frac{R^2}{r^2} + p \right) \right\} - \left(\frac{r^2}{R^2} + p \right) \right] v(r) dr = 0 \qquad \forall v(r)$$

2.4 Hence show that the radii of the tube after deformation must satisfy the equations

$$\log(B^2 / a^2) + \frac{B^2}{a^2} = \log(A^2 / b^2) + \frac{A^2}{b^2}$$
$$b^2 - a^2 = B^2 - A^2$$

3. The goal of this problem is to derive an expression for the maximum height reached by a rubber Helium balloon that is released from the surface of the earth. Make the following assumptions:



- The balloon has total mass (the rubber, plus the He inside) m
- The balloon is a thin walled sphere, and prior to inflation has wall thickness t_0 and radius a_0.
- At the earth's surface the balloon has radius a_1 and internal pressure p_{b0}
- Both He and the air can be idealized as ideal gases
- The balloon can be idealized as an incompressible, neo-Hookean solid with Cauchy stress-stretch relation $\boldsymbol{\sigma} = \mu \mathbf{B} + p\mathbf{I}$
- The air temperature θ is constant (i.e does not vary with altitude), and the balloon is always in thermal equilibrium with the air.

3.1 Given that the air pressure has magnitude p_{a0} at the earth's surface, calculate the variation of air pressure p_a and density ρ_a with height z above the earth's surface, in terms of gravitational acceleration g, (constant) temperature θ and the gas constant for air R_a.

3.2 Show that the radius a for which the balloon is neutrally buoyant is related to air pressure p_a by

$$a = \left(\frac{3mR_a\theta}{4\pi p_a} \right)^{1/3}$$

3.3 Consider equilibrium of the thin walled spherical balloon. Using the thin-walled pressure vessel approximation, find an expression for the Cauchy stress components $\sigma_{\phi\phi}, \sigma_{\theta\theta}$ in the balloon, in terms of the internal pressure in the balloon p_b, the external air pressure p_a, the deformed wall thickness t and the radius of the balloon a. (you can assume $\sigma_{rr} \approx 0$)

3.4 Write down the components of deformation gradient in the balloon (in spherical polar coordinates), in terms of the initial and deformed wall thicknesses t_0, t, and the undeformed and deformed radii of the balloon a_0, a. Neglect variations through the thickness of the wall.

3.5 Use the incompressibility condition to relate t to t_0, a_0, a.

3.6 Use the constitutive equation to find an expression for the Cauchy stress $\sigma_{\theta\theta}$ in terms of μ, a, a_0. Assume $\sigma_{rr} = 0$ and neglect variations through the thickness of the wall.

3.7 Given that the pressure in the balloon is p_{b0} at the surface of the earth, and the balloon has radius a_1 at the surface of the earth, show that at altitude *z* the internal pressure in the balloon is related to its radius *a* by

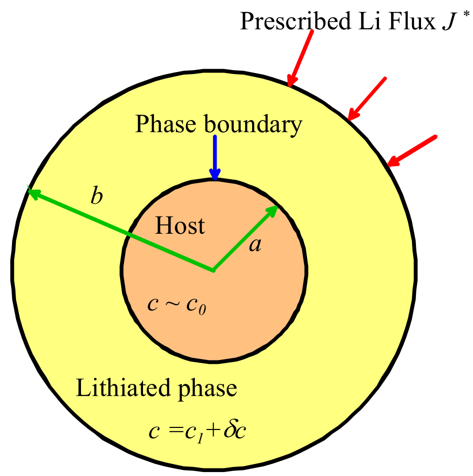
$$p_b = \frac{a_1^3}{a^3} p_{b0}$$

3.8 Hence, show that the radius of the neutrally buoyant balloon satisfies the equation

$$\left(p_{b0} - \frac{3mR_a\theta}{4\pi a_1^3} \right) = \frac{2\mu t_0}{a_1} \left(\frac{a}{a_1} \right)^2 \left\{ 1 - \left(\frac{a_0}{a} \right)^6 \right\}$$

3.9 Assuming that $(a_0 / a)^6 << 1$, find an expression for the altitude of the balloon when it is neutrally buoyant.

4. The figure shows a spherical Li-ion battery particle. The host material has a negligible Li concentration, and Li is inserted into the particle through an electrochemical reaction at the particle surface. The material phase separates with equilibrium concentrations c_0, c_1 so that at some representative instant during Li insertion the particle consists of a spherical core with radius a containing a low uniform Li concentration c_0 surrounded by an outer shell with higher, nonuniform Li concentration $c_1 + \delta c$, where δc is to be determined.



The material is elastic with (concentration independent) Young's modulus E and Poisson ratio ν. When lithiated, the material experiences a compositional strain $\varepsilon_{ij}^c = \beta c \delta_{ij} / 3$, where β is a constant (quantifying the volumetric strain caused by Li insertion), so the total strain in the sphere (taking the un-lithiated material as reference configuration) is

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \frac{\beta c}{3}\delta_{ij}$$

The deviation of Li concentration from its equilibrium value δc in the sphere satisfies (approximately) the diffusion equations

$$\nabla \cdot (\nabla \mu) = 0 \qquad a < r < b \qquad \mu = \mu_0 + \Gamma \delta c - \frac{1}{3}\beta \mathrm{trace}(\boldsymbol{\sigma})$$

$$\delta c = 0 \qquad r = a$$

$$D\frac{\partial \mu}{\partial r} = J^* \qquad r = b$$

where μ_0, Γ are constants, and D is the diffusion coefficient for Li transport through the outer shell.

4.1 Assume that the state of stress in the core region $0<r<a$ is a state of uniform hydrostatic stress $\sigma_{rr} = \sigma_{\theta\theta} = p$. Calculate the displacement field (relative to a sphere with zero Li concentration) in the core region, in terms of p and other relevant variables.

4.2 Show that the displacement field in $a<r<b$ must satisfy

$$\left.\begin{aligned} \frac{du_r}{dr} + \frac{2u_r}{r} &= \beta c_1 + \beta \delta c + \frac{1-2\nu}{E}(\sigma_{rr} + 2\sigma_{\theta\theta}) \\ \frac{du_r}{dr} - \frac{u_r}{r} &= \left(\frac{1+\nu}{E}(\sigma_{rr} - \sigma_{\theta\theta})\right) \end{aligned}\right\} \qquad a < r < b$$

4.3 Show (use a symbolic manipulation program to do the algebra) that the equation of equilibrium can be expressed in the form

$$\frac{d^2u_r}{dr^2}+\frac{2}{r}\frac{du_r}{dr}-\frac{2}{r^2}u_r=\frac{\beta(1+\nu)}{3(1-\nu)}\frac{dc}{dr}$$

4.4 Solve 5.3 for the displacements (which will include some arbitrary constants that need not be determined), and hence show that the diffusion equation can be expressed in the form

$$\left(\frac{2E\beta^2}{9(1-\nu)}+\Gamma\right)\frac{d\delta c}{dr}=\frac{J^*b^2}{Dr^2}$$

4.5 Hence show that the Li concentration in $a<r<b$ is

$$c=c_1+\frac{Jb^2}{D\left[\Gamma+2E\beta^2/9(1-\nu)\right]}\frac{r-a}{ar}$$

4.6 Finally, show that the stress field in the particle is

$$\sigma_{rr}=\frac{2E\beta(c_1-c_0)}{9(1-\nu)}\frac{a^3(b^3-r^3)}{b^3r^3}-\frac{E\beta J^*}{9D(1-\nu)\Lambda}\frac{(b-r)}{br^3}(a^2b^2+a^2br+a^2r^2-3b^2r^2)$$

$$\sigma_{\theta\theta}=\frac{2E\beta(c_1-c_0)}{9(1-\nu)}\frac{a^3(b^3+2r^3)}{b^3r^3}+\frac{E\beta J^*}{18D(1-\nu)\Lambda}\frac{(a^2b^3+2a^2r^3+3b^3r^2-6b^2r^3)}{br^3}$$

$$\Lambda=\frac{2E\beta^2}{9(1-\nu)}+\Gamma$$

To do this you will need to (i) Substitute 5.5 into 5.3, solve the ODE, and determine the unknown constants of integration from the boundary condition for radial stress at $r=b$ and continuity of stress and displacement at $r=a$. The algebra is tiresome and best done with a symbolic manipulation program.