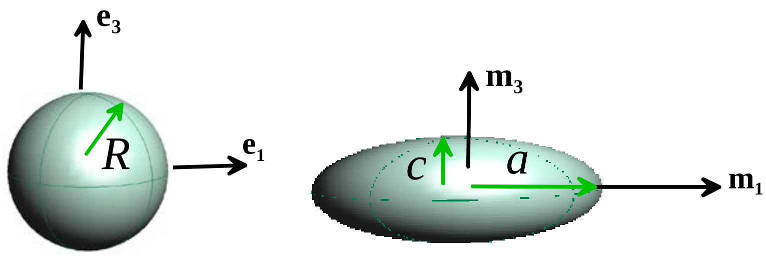
School of Engineering
Brown University

EN2210: Continuum Mechanics

Homework 8: Elasticity
Solutions



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1. An incompressible neo-hookean sphere with radius *R*, modulus μ and mass density ρ is deformed into an ellipsoid with semi-axes $a = \lambda_1 R \quad b = \lambda_1 R \quad c = \lambda_2 R$.

1.1 Use the incompressibility condition to express λ_2 in terms of λ_1

Incompressibility requires $\lambda_1^2 \lambda_2 = 1 \Rightarrow \lambda_2 = \dfrac{1}{\lambda_1^2}$

[2 POINTS]

1.2 Find an expression for the total elastic strain energy in the solid, in terms of λ_1

We have that $\mathbf{B} = \mathbf{F}\mathbf{F}^T = \begin{bmatrix} \lambda_1^2 & & \\ & \lambda_1^2 & \\ & & 1/\lambda_1^4 \end{bmatrix}$

The strain energy density is $\psi = \dfrac{\mu}{2}(tr\mathbf{B} - 3) = \dfrac{\mu}{2}\left(2\lambda_1^2 + \dfrac{1}{\lambda_1^4} - 3\right)$ and the total strain energy is

$$U = \frac{2\pi R^3 \mu}{3}\left(2\lambda_1^2 + \frac{1}{\lambda_1^4} - 3\right)$$

[2 POINTS]

1.3 Find an expression for the total kinetic energy of the solid, in terms of λ_1 and its time derivatives. Assume the center is stationary.

The velocity field is $v_1 = \dot{\lambda}_1 x_1 \qquad v_2 = \dot{\lambda}_1 x_2 \qquad v_3 = \dot{\lambda}_3 x_3 = -2\frac{\dot{\lambda}_1}{\lambda_1^3} x_3$. The kinetic energy follows as

$$T = \frac{\rho}{2}\int_V \dot{\lambda}_1^2 \left(x_1^2 + x_2^2\right) + 4\frac{\dot{\lambda}_1^2}{\lambda_1^6} x_3^2 dV$$

where the integral is taken over the undeformed sphere.

Note that $\int_V x_i x_j dV = \frac{4\pi}{15} R^5 \delta_{ij}$ so the KE follows as

$$T = \frac{4\pi R^5 \rho}{15} \dot{\lambda}_1^2 \left(1 + \frac{2}{\lambda_1^6}\right)$$

[2 POINTS]

1.4 Hence, estimate the natural frequency of small amplitude oscillations of this (approximate) vibration mode

We can derive an equation of motion from energy conservation:

$$U + T = \text{constant}$$

$$\Rightarrow \frac{d}{dt}(U+T) = \frac{8\pi R^4 \mu}{3}\dot{\lambda}_1\left(\lambda_1 - \frac{1}{\lambda_1^5}\right) + \frac{8\pi R^5 \rho}{15}\left(\dot{\lambda}_1\ddot{\lambda}_1 + 2\frac{\dot{\lambda}_1\ddot{\lambda}}{\lambda^6} - \frac{12\dot{\lambda}_1^3}{\lambda_1^7}\right) = 0$$

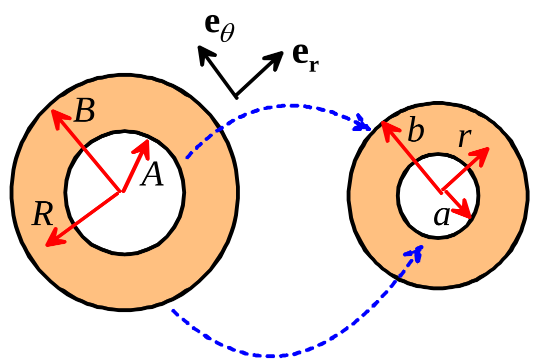
$$\Rightarrow \frac{3R^2\rho}{5}\ddot{\lambda}_1 + \mu\left(\lambda_1 - \frac{1}{\lambda_1^5}\right) \approx 0$$

Set $\lambda_1 = 1 + \varepsilon$

$$\Rightarrow \frac{3R^2\rho}{5}\ddot{\varepsilon} + 6\mu\varepsilon = 0 \Rightarrow \omega = \sqrt{\frac{10\mu}{R^2\rho}}$$

[2 POINTS]

2. A rubber tube with internal radius A and external radius B is turned inside out, so that the surfaces at $R=A$, $R=B$ now lie at deformed radius $r=b$, $r=a$, respectively. These surfaces are free of traction. Assume that plane cross-sections of the tube remain plane, ***and that the length of the cylinder does not change***. Note that a cross section at distance Z along the axis of the undeformed tube moves to a new position $z=-Z$ after deformation. The tube may be idealized as an incompressible, neo-Hookean material with stress-strain relation



$$\boldsymbol{\sigma} = \mu \mathbf{B} + p(r)\mathbf{I}$$

2.1 By considering the volumes of material in the annular regions between r and a, and between R and B, find an expression for the radial position r of a material particle that starts at radius R in the tube before deformation.

The volume of material outside radius r must match the volume in side radius R, and vice-versa. This requires that

$$r^2 - a^2 = B^2 - R^2$$

[2 POINTS]

2.2 Show that the deformation gradient $\mathbf{F}$ is given by

$$\mathbf{F} = \begin{bmatrix} -\dfrac{R}{r} & 0 & 0 \\ 0 & \dfrac{r}{R} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and hence write down an expression for the stress in the tube as a function of r, R and p.

The deformation gradient is

$$\mathbf{F} = \begin{bmatrix} \dfrac{dr}{dR} & 0 & 0 \\ 0 & \dfrac{r}{R} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The condition det($\mathbf{F}$)=1 shows that $dr/dR = -R/r$, and $\mathbf{F}$ follows. Furthermore, it follows that

$$\mathbf{B} = \begin{bmatrix} \frac{R^2}{r^2} & 0 & 0 \\ 0 & \frac{r^2}{R^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The stress is therefore

$$\boldsymbol{\sigma} = \begin{bmatrix} \mu\frac{R^2}{r^2} + p & 0 & 0 \\ 0 & \mu\frac{r^2}{R^2} + p & 0 \\ 0 & 0 & \mu + p \end{bmatrix}$$

[3 POINTS]

2.3 By considering a virtual velocity $\delta\mathbf{v} = v(r)\mathbf{e}_r$ where $v(r)$ is a continuously differentiable function, show that the principle of virtual work requires that

$$\int_a^b \left[\frac{d}{dr}\left\{ r\left(\frac{R^2}{r^2} + p \right) \right\} - \left(\frac{r^2}{R^2} + p \right) \right] v(r)dr = 0 \qquad \forall v(r)$$

The virtual work equation requires that

$$\int_V \boldsymbol{\sigma} : \nabla\delta\mathbf{v} dV + \int_B \mathbf{n}\cdot\boldsymbol{\sigma}\cdot\delta\mathbf{v} dA = 0$$

The virtual velocity gradient in cylindrical coordinates is

$$\nabla\delta\mathbf{v} = \frac{d\delta v}{dr}\mathbf{e}_r \otimes \mathbf{e}_r + \frac{\delta v}{r}\mathbf{e}_\theta \otimes \mathbf{e}_\theta$$

Substituting this, and the stress, into the virtual work equation gives

$$\int_a^b \left\{ \left(\mu\frac{R^2}{r^2} + p \right)\frac{d\delta v}{dr} + \left(\mu\frac{r^2}{R^2} + p \right)\frac{\delta v}{r} \right\} 2\pi r dr = 0$$

Integrating the first term by parts yields

$$\int_a^b \left\{ -\frac{d}{dr} r\left(\mu\frac{R^2}{r^2} + p \right) + \left(\mu\frac{r^2}{R^2} + p \right) \right\} \delta v dr = 0$$

[5 POINTS]

2.4 Hence show that the radii of the tube after deformation must satisfy the equations

$$\log(B^2/a^2) + \frac{B^2}{a^2} = \log(A^2/b^2) + \frac{A^2}{b^2}$$
$$b^2 - a^2 = B^2 - A^2$$

The previous problem shows that p satisfies the equation

$$r\frac{dp}{dr} = -\mu\frac{d}{dr}\left(\frac{R^2}{r}\right) + \mu\frac{r^2}{R^2}$$

$$\Rightarrow \frac{1}{\mu}\frac{dp}{dr} = -\frac{1}{r}\frac{d}{dr}\left(\frac{r^2 - a^2 + B^2}{r}\right) + \frac{r}{r^2 - a^2 + B^2} = -\frac{1}{r} + \frac{B^2 - a^2}{r^3} + \frac{r}{r^2 - a^2 + B^2}$$

$$\Rightarrow \frac{p}{\mu} = -\log(r) + \log(r^2 - a^2 + B^2) - \frac{B^2 - a^2}{2r^2} + C$$

$$\Rightarrow \frac{p}{\mu} = \log(R/r) - \frac{r^2 + B^2 - a^2}{2r^2} + D = \log(R/r) - \frac{R^2}{2r^2} + D$$

The boundary conditions are

$$\frac{\sigma_{rr}}{\mu} = \frac{p}{\mu} + \frac{R^2}{r^2} = 0 \qquad (R = B, r = a) \qquad (R = A, r = b)$$

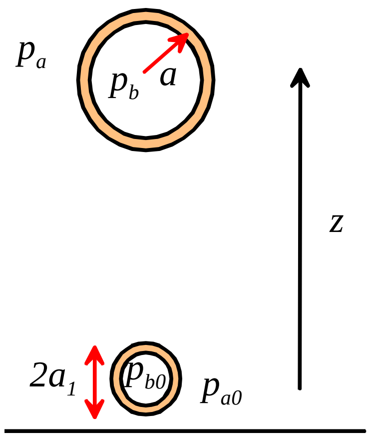
This yields

$$\log(B/a) + \frac{B^2}{2a^2} + D = 0 \qquad \log(A/b) + \frac{A^2}{2b^2} + D = 0$$

$$\Rightarrow \log(B/a) + \frac{B^2}{2a^2} = \log(A/b) + \frac{A^2}{2b^2}$$

[5 POINTS]

3. The goal of this problem is to derive an expression for the maximum height reached by a rubber Helium balloon that is released from the surface of the earth. Make the following assumptions:



- The balloon has total mass (the rubber, plus the He inside) m
- The balloon is a thin walled sphere, and prior to inflation has wall thickness t_0 and radius a_0.
- At the earth's surface the balloon has radius a_1 and internal pressure p_{b0}
- Both He and the air can be idealized as ideal gases
- The balloon can be idealized as an incompressible, neo-Hookean solid with Cauchy stress-stretch relation $\boldsymbol{\sigma} = \mu \mathbf{B} + p\mathbf{I}$
- The air temperature θ is constant (i.e does not vary with altitude), and the balloon is always in thermal equilibrium with the air.

3.1 Given that the air pressure has magnitude p_{a0} at the earth's surface, calculate the variation of air pressure p_a and density ρ_a with height z above the earth's surface, in terms of gravitational acceleration g, (constant) temperature θ and the gas constant for air R_a.

The ideal gas equation gives $p_a = \rho_a R_a \theta$

The equilibrium equation gives $-\frac{\partial p_a}{\partial z} - \rho_a g = 0$

Combining these gives
$$\frac{\partial p_a}{\partial z} = -\frac{g p_a}{R_a \theta} \Rightarrow p_a = p_{a0} \exp(-gz / R_a \theta)$$
$$\rho_a = \frac{p_0}{R_a \theta} \exp(-gz / R_a \theta) = \rho_0 \exp(-gz / R_a \theta)$$

[3 POINTS]

3.2 Show that the radius a for which the balloon is neutrally buoyant is related to air pressure p_a by

$$a = \left(\frac{3mR_a\theta}{4\pi p_a} \right)^{1/3}$$

For neutral buoyancy: $m = \frac{4}{3}\pi a^3 \rho_a \Rightarrow a = \left(\frac{3m}{4\pi \rho_a} \right)^{1/3}$

The gas law then gives $a = \left(\frac{3mR_a\theta}{4\pi p_a} \right)^{1/3}$

[2 POINTS]

3.3 Consider equilibrium of the thin walled spherical balloon. Using the thin-walled pressure vessel approximation, find an expression for the Cauchy stress components $\sigma_{\phi\phi}, \sigma_{\theta\theta}$ in the balloon, in terms of the internal pressure in the balloon p_b, the external air pressure p_a, the deformed wall thickness *t* and the radius of the balloon *a*. (you can assume $\sigma_{rr} \approx 0$)

Equilibrium of a diametrical cross section gives

$$\pi a^2 (p_b - p_a) = 2\pi a t \sigma_{\theta\theta} \Rightarrow \sigma_{\theta\theta} = \frac{(p_b - p_a) a}{2t}$$

[2 POINTS]

3.4 Write down the components of deformation gradient in the balloon (in spherical polar coordinates), in terms of the initial and deformed wall thicknesses t_0, t, and the undeformed and deformed radii of the balloon a_0, a. Neglect variations through the thickness of the wall.

$$\mathbf{F} = \begin{bmatrix} t/t_0 & 0 & 0 \\ 0 & a/a_0 & 0 \\ 0 & 0 & a/a_0 \end{bmatrix}$$

[2 POINTS]

3.5 Use the incompressibility condition to relate *t* to t_0, a_0, a .

$$J=1 \text{ gives } \frac{t}{t_0}\frac{a^2}{a_0^2} = 1 \Rightarrow t = t_0 \frac{a_0^2}{a^2}$$

[1 POINT]

3.6 Use the constitutive equation to find an expression for the Cauchy stress $\sigma_{\theta\theta}$ in terms of μ, a, a_0. Assume $\sigma_{rr} = 0$ and neglect variations through the thickness of the wall.

We have that

$$\boldsymbol{\sigma} = \begin{bmatrix} \mu(a_0/a)^4 + p & 0 & 0 \\ 0 & \mu(a/a_0)^2 + p & 0 \\ 0 & 0 & \mu(a/a_0)^2 + p \end{bmatrix} \Rightarrow \sigma_{\theta\theta} = \mu\left\{\left(\frac{a}{a_0}\right)^2 - \left(\frac{a_0}{a}\right)^4\right\}$$

[2 POINTS]

3.7 Given that the pressure in the balloon is p_{b0} at the surface of the earth, and the balloon has radius a_1 at the surface of the earth, show that at altitude z the internal pressure in the balloon is related to its radius a by

$$p_b = \frac{a_1^3}{a^3} p_{b0}$$

Mass conservation gives $\rho_{b0} a_1^3 = \rho_b a^3$, and the ideal gas law gives $p_{b0} = \rho_{b0} R_b \theta \quad p_b = \rho_b R_b \theta$. Since temperature is constant pressure and density are directly proportional, which gives the answer stated.

[2 POINTS]

3.8 Hence, show that the radius of the neutrally buoyant balloon satisfies the equation

$$\left(p_{b0} - \frac{3mR_a\theta}{4\pi a_1^3} \right) = \frac{2\mu t_0}{a_1} \left(\frac{a}{a_1} \right)^2 \left\{ 1 - \left(\frac{a_0}{a} \right)^6 \right\}$$

Combining 3.3 and 3.6 shows that $\dfrac{(p_b - p_a)a}{2t} = \mu \left\{ \left(\dfrac{a}{a_0} \right)^2 - \left(\dfrac{a_0}{a} \right)^4 \right\}$

Substitute for p_b from 3.7 , t from 3.5 and p_a from 3.2 gives

$$\frac{\left(\dfrac{a_1^3}{a^3} p_{b0} - \dfrac{3mR_a\theta}{4\pi a^3} \right) a}{2t_0 \left(\dfrac{a_0}{a} \right)^2} = \mu \left\{ \left(\frac{a}{a_0} \right)^2 - \left(\frac{a_0}{a} \right)^4 \right\}$$

This expression can be rearranged as

$$\left(p_{b0} - \frac{3mR_a\theta}{4\pi a_1^3} \right) = \frac{2\mu t_0}{a_1} \left(\frac{a}{a_1} \right)^2 \left\{ 1 - \left(\frac{a_0}{a} \right)^6 \right\}$$

[4 POINTS]

3.9 Assuming that $(a_0 / a)^6 << 1$, find an expression for the altitude of the balloon when it is neutrally buoyant.

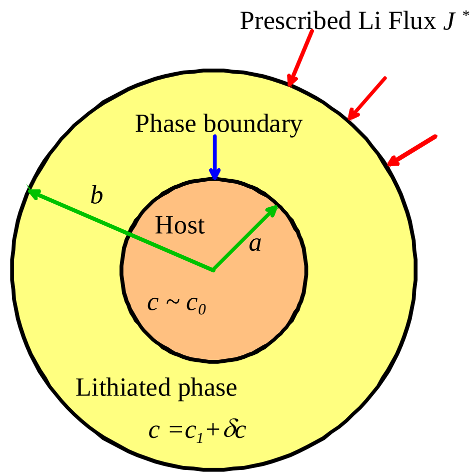
From 3.2 we have that

$$\frac{p_a}{p_{a0}} = \frac{3mR_a\theta}{4\pi\, p_{a0} a^3} = \frac{3m}{4\pi\rho_{a0} a_1^3} \left(\frac{a_1}{a} \right)^3 = \frac{3m}{4\pi\rho_{a0} a_1^3} \left(\frac{a_1}{2\mu t_0} \right)^{3/2} \left(p_{b0} - \frac{3mR_a\theta}{4\pi a_1^3} \right)^{-3/2}$$

$$\Rightarrow z = \frac{R_a\theta}{g} \log \left[\frac{4\pi\rho_{a0} a_1^3}{3m} \left(\frac{2\mu t_0}{a_1} \right)^{3/2} \left(p_{b0} - \frac{3mR_a\theta}{4\pi a_1^3} \right)^{3/2} \right]$$

[2 POINTS]

4. The figure shows a spherical Li-ion battery particle. The host material has a negligible Li concentration, and Li is inserted into the particle through an electrochemical reaction at the particle surface. The material phase separates with equilibrium concentrations c_0, c_1 so that at some representative instant during Li insertion the particle consists of a spherical core with radius a containing a low uniform Li concentration c_0 surrounded by an outer shell with higher, nonuniform Li concentration $c_1 + \delta c$, where δc is to be determined.



The material is elastic with (concentration independent) Young's modulus E and Poisson ratio ν. When lithiated, the material experiences a compositional strain $\varepsilon_{ij}^c = \beta c \delta_{ij} / 3$, where β is a constant (quantifying the volumetric strain caused by Li insertion), so the total strain in the sphere (taking the un-lithiated material as reference configuration) is

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \frac{\beta c}{3}\delta_{ij}$$

The deviation of Li concentration from its equilibrium value δc in the sphere satisfies (approximately) the diffusion equations

$$\nabla \cdot (\nabla \mu) = 0 \qquad a < r < b \qquad \mu = \mu_0 + \Gamma \delta c - \frac{1}{3}\beta \text{trace}(\boldsymbol{\sigma})$$

$$\delta c = 0 \qquad r = a$$

$$D\frac{\partial \mu}{\partial r} = J^* \qquad r = b$$

where μ_0, Γ are constants, and D is the diffusion coefficient for Li transport through the outer shell.

4.1 Assume that the state of stress in the core region $0<r<a$ is a state of uniform hydrostatic stress $\sigma_{rr} = \sigma_{\theta\theta} = p$. Calculate the displacement field (relative to a sphere with zero Li concentration) in the core region, in terms of p and other relevant variables.

The total strains follow as $\varepsilon_{rr} = \varepsilon_{\theta\theta} = \frac{1-2\nu}{E}p + \frac{\beta}{3}c_0$ and hence the displacement field is

$$u_r = r\left(\frac{1-2\nu}{E}p + \frac{\beta}{3}c_0\right)$$

[2 POINTS]

4.2 Show that the displacement field in $a<r<b$ must satisfy

$$\left.\begin{aligned} \frac{du_r}{dr}+\frac{2u_r}{r} &= \beta c_1+\beta\delta c+\frac{1-2\nu}{E}(\sigma_{rr}+2\sigma_{\theta\theta}) \\ \frac{du_r}{dr}-\frac{u_r}{r} &= \left(\frac{1+\nu}{E}(\sigma_{rr}-\sigma_{\theta\theta})\right) \end{aligned}\right\} \qquad a<r<b$$

The strain-displacement relations in spherical coordinates are

$\varepsilon_{rr}=\frac{\partial u_r}{\partial r} \qquad \varepsilon_{\theta\theta}=\frac{u_r}{r}$. The elastic stress-strain relations give

$$\varepsilon_{rr}+2\varepsilon_{\theta\theta}=\frac{1-2\nu}{E}(\sigma_{rr}+2\sigma_{\theta\theta})+\beta c\delta_{ij}$$

$$\varepsilon_{rr}-\varepsilon_{\theta\theta}=\frac{1+\nu}{E}(\sigma_{rr}-\sigma_{\theta\theta})$$

Combining these gives the result stated.

[2 POINTS]

4.3 Show (use a symbolic manipulation program to do the algebra) that the equation of equilibrium can be expressed in the form

$$\frac{d^2u_r}{dr^2}+\frac{2}{r}\frac{du_r}{dr}-\frac{2}{r^2}u_r=\frac{\beta(1+\nu)}{3(1-\nu)}\frac{dc}{dr}$$

The equilibrium equation in spherical coordinates is

$$\frac{d\sigma_{rr}}{dr}+\frac{2}{r}(\sigma_{rr}-\sigma_{\theta\theta})=0$$

Equations in 4.2, can be solved for the stresses and substituted into this equation to obtain the desired result (see the mupad code for details)

[4 POINTS]

4.4 Solve 4.3 for the displacements (which will include some arbitrary constants that need not be determined), and hence show that the diffusion equation can be expressed in the form

$$\left(\frac{2E\beta^2}{9(1-\nu)}+\Gamma\right)\frac{d\delta c}{dr}=\frac{J^*b^2}{Dr^2}$$

Note that the diffusion equation in spherical coordinates is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\mu}{dr}\right)=0$$

Together with the boundary condition at the surface this yields

$$\frac{d\mu}{dr}=-\frac{J^*a^2}{Dr^2}$$

Solve 4.3, then substitute into 4.2 and solve for the stresses, then substitute these into the diffusion equation and simplify. See the mupad code for details.

[4 POINTS]

4.5 Hence show that the Li concentration in $a<r<b$ is

$$c = c_1 + \frac{Jb^2}{D\left[\Gamma + 2E\beta^2 / 9(1-\nu)\right]}\frac{r-a}{ar}$$

The equation in 4.4 can be solved with boundary condition $\delta c = 0 \quad r = a$. See the mupad code for details.

[2 POINTS]

4.6 Finally, show that the stress field in the particle is

$$\sigma_{rr} = \frac{2E\beta(c_1 - c_0)}{9(1-\nu)}\frac{a^3(b^3 - r^3)}{b^3 r^3} - \frac{E\beta J^*}{9D(1-\nu)\Lambda}\frac{(b-r)}{br^3}(a^2b^2 + a^2br + a^2r^2 - 3b^2r^2)$$

$$\sigma_{\theta\theta} = \frac{2E\beta(c_1 - c_0)}{9(1-\nu)}\frac{a^3(b^3 + 2r^3)}{b^3 r^3} + \frac{E\beta J^*}{18D(1-\nu)\Lambda}\frac{(a^2b^3 + 2a^2r^3 + 3b^3r^2 - 6b^2r^3)}{br^3}$$

$$\Lambda = \frac{2E\beta^2}{9(1-\nu)} + \Gamma$$

To do this you will need to (i) Substitute 4.5 into 4.3, solve the ODE, and determine the unknown constants of integration from the boundary condition for radial stress at $r=b$ and continuity of stress and displacement at $r=a$. The algebra is tiresome and best done with a symbolic manipulation program.

See the Mupad code for the solution.

[5 POINTS]