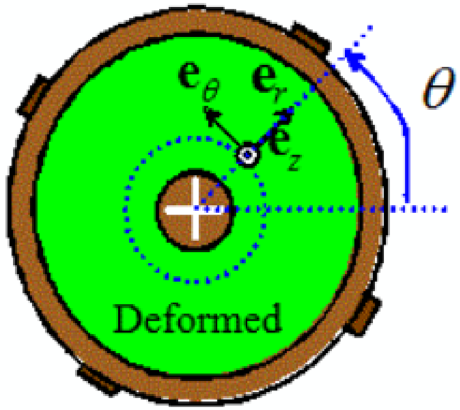
Review

Viscous fluid in cylindrical shear cell



Velocity gradient $\mathbf{L} = \nabla_{\mathbf{y}} \mathbf{v} = \frac{\partial v_\theta}{\partial r} \mathbf{e}_\theta \otimes \mathbf{e}_r - \frac{v_\theta}{r} \mathbf{e}_r \otimes \mathbf{e}_\theta$

Stretch rate $\mathbf{D} = sym(\mathbf{L}) = \frac{1}{2}\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right)(\mathbf{e}_\theta \otimes \mathbf{e}_r + \mathbf{e}_r \otimes \mathbf{e}_\theta)$

Acceleration $\mathbf{a} = \left. \frac{\partial \mathbf{v}}{\partial t} \right|_{\mathbf{y}} + \mathbf{L}\mathbf{v} = -\frac{v^2}{r} \mathbf{e}_r$

Stress $\boldsymbol{\tau} = \boldsymbol{\sigma} = 2\mu \mathbf{D} + p\mathbf{I} = \mu \left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right)(\mathbf{e}_\theta \otimes \mathbf{e}_r + \mathbf{e}_r \otimes \mathbf{e}_\theta) + p\mathbf{I}$

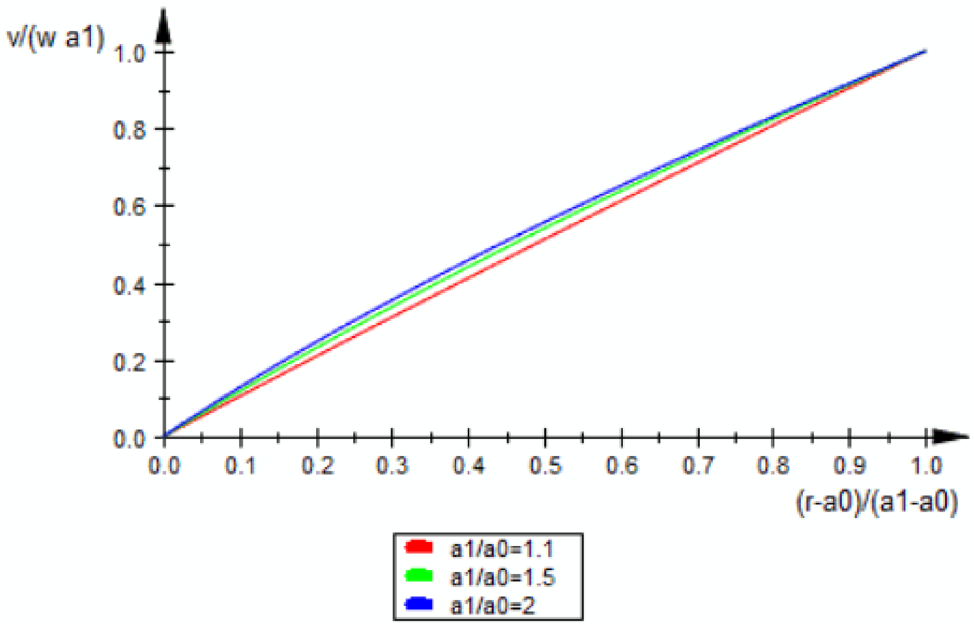
Linear Momentum $\nabla_{\mathbf{y}} \cdot \boldsymbol{\sigma} = \rho \mathbf{a} = \mu \frac{\partial}{\partial r}\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right)\mathbf{e}_\theta + 2\mu \left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right)\mathbf{e}_\theta + \frac{\partial p}{\partial r} \mathbf{e}_r = -\rho \frac{v^2}{r} \mathbf{e}_r$

Solve, with BCs $\mathbf{v}(a_0) = 0 \qquad \mathbf{v}(a_1) = a_1 \omega \mathbf{e}_\theta$

```
diffeq1 := diff(diff(vq(r),r),r) + diff(vq(r),r)/r - vq(r)/r^2=0:
bc := vq(a0)=0,vq(a1)=a1*w:
vqsol := simplify(solve(ode({diffeq1,bc}, vq(r)),IgnoreSpecialCases))[1]
```

$$\frac{a1^2\, w\, (a0^2 - r^2)}{r\, (a0^2 - a1^2)}$$

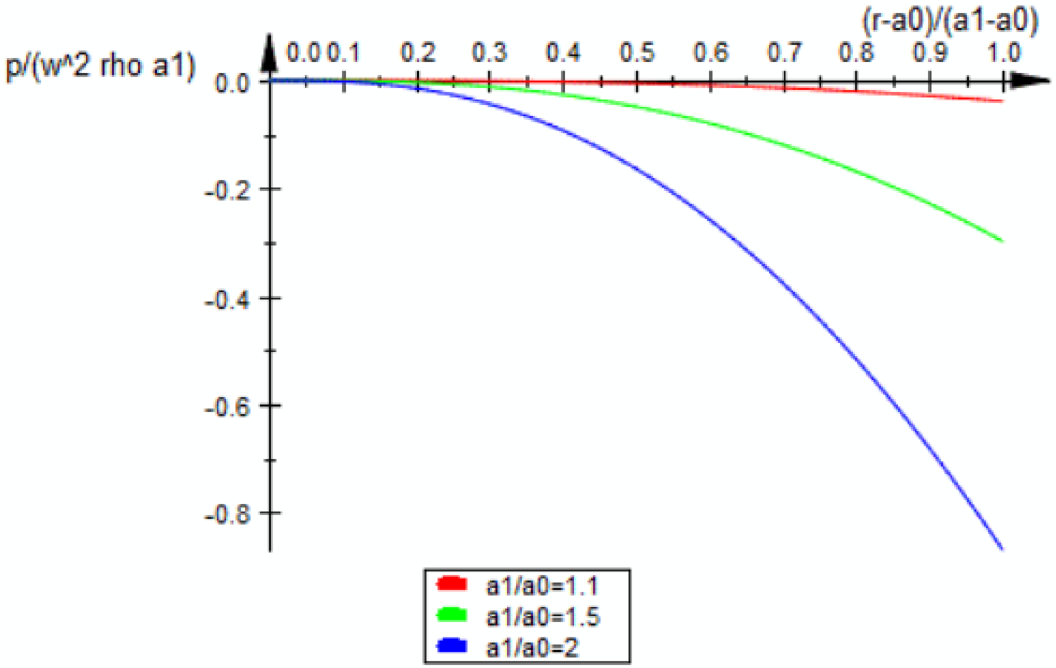
```
vqnorm := subs(vqsol/a1,{a0=1,w=1,r=1+(a1-1)*xi}):
plot(
plot::Function2d(subs(vqnorm,a1=1.1),xi=0..1,Color=RGB::Red,Legend="a1/a0=1.1"),
plot::Function2d(subs(vqnorm,a1=1.5),xi=0..1,Color=RGB::Green,Legend="a1/a0=1.5"),
plot::Function2d(subs(vqnorm,a1=2),xi=0..1,Color=RGB::Blue,Legend="a1/a0=2"),
AxesTitles = ["(r-a0)/(a1-a0)","v/(w a1)"]
)
```



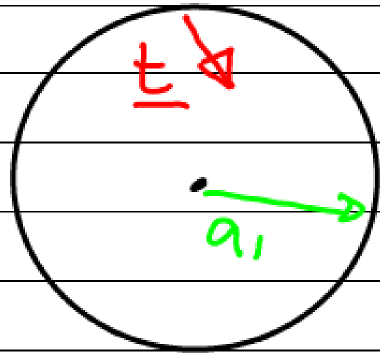
```
diffeq2 := diff(p(r),r) = -rho*vqsol^2/r:
psol := simplify(solve(ode({diffeq2}, p(r)),IgnoreSpecialCases))[1]
```

$$C5 + \frac{a1^4\, rho\, w^2\, (a0^4 - r^4 + 4\, a0^2\, r^2 \ln(r))}{2\, r^2\, (a0^2 - a1^2)^2}$$

```
pnorm := subs(psol,{C5=0,rho=1,a0=1,w=1,r=1+(a1-1)*xi}):
plot(
plot::Function2d(subs(pnorm,a1=1.1),xi=0..1,Color=RGB::Red,Legend="a1/a0=1.1"),
plot::Function2d(subs(pnorm,a1=1.5),xi=0..1,Color=RGB::Green,Legend="a1/a0=1.5"),
plot::Function2d(subs(pnorm,a1=2),xi=0..1,Color=RGB::Blue,Legend="a1/a0=2"),
AxesTitles = ["(r-a0)/(a1-a0)","p/(w^2 rho a1)"])
```



(f) Find an expression for the torque (per unit out of plane distance) necessary to rotate the external cylinder



Traction exerted on cylinder is equal & opposite to traction acting on fluid

Resultant moment on fluid :

$$\int_0^{2\pi} \underline{r} \times \underline{t} \; a_1 \, d\theta \qquad \underline{t} = \underline{n} \cdot \underline{\sigma}$$

$$\underline{\sigma} = \mu \left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right) \left(\underline{e_r} \otimes \underline{e_\theta} + \underline{e_\theta} \otimes \underline{e_r} \right) + p\underline{I}$$

$$\underline{n} = \underline{e_r}$$

$$\underline{t} = p \, \underline{e_r} + \mu \left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right) \underline{e_\theta}$$

$$a_1 \underline{e_r} \times \underline{t} = \mu a_1 \left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right) \underline{e_z}$$

```
torque := simplify(subs(2*PI*a1^2*`&mu;`*(diff(vqsol,r)-vqsol/r),r=a1))
```

$$-\frac{4\pi\mu\, a0^2\, a1^2\, w}{a0^2 - a1^2} = Q$$

(g) Calculate the rate of external work done by the torque acting on the rotating exterior cylinder

$$W = Q\omega$$

(h) Calculate the rate of internal dissipation in the solid as a function of *r*.

$$\text{Dissipation} \quad r_p = \sigma : D \qquad \sigma = 2\mu D + pI$$

$$= 2\mu\, D:D + p\,\underbrace{I:D}_{tr(D)}$$

$$r_p = 2\mu\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)^2$$

```
dissip := simplify(`&mu;`*(diff(vqsol,r)-vqsol/r)^2)
```

$$\frac{4\mu\, a0^4\, a1^4\, w^2}{r^4\,(a0^2 - a1^2)^2}$$

(i) Show that the total internal dissipation is equal to the rate of work done by the external moment.

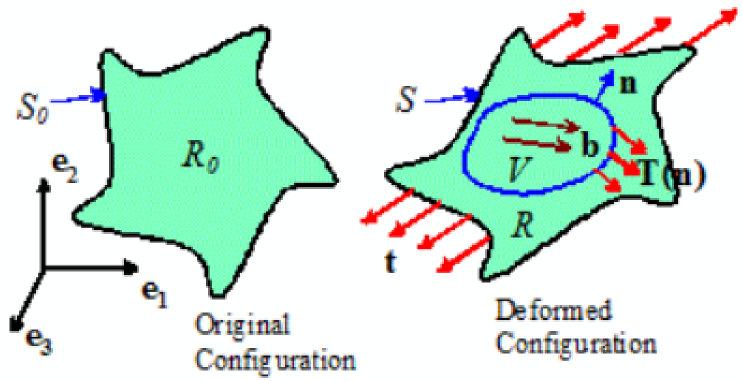
Total dissipation $= \int_{a_0}^{a_1} r_p \, 2\pi r \, dr$

```
assume(a1>a0): assume(a1>0): assume(a0>0):
int(dissip*2*PI*r,r=a0..a1)
```

$$-\frac{4\,\pi\,\mu\,a0^2\,a1^2\,w^2}{a0^2-a1^2}$$

Principle of "Virtual Work"

- "Weak form" of LMB
- Basis for FEA



Let σ be a symmetric tensor on V

σ satisfies

$$\left.\begin{aligned} \nabla_y \cdot \sigma + \rho \underline{b} &= \rho \left.\frac{\partial \underline{v}}{\partial t}\right|_{\underline{x}} \\ \underline{t} &= n \cdot \sigma \end{aligned}\right\} \quad 1$$

Define "Virtual Velocity" field $\delta \underline{v}(\underline{y})$

- arbitrary vector field on V that is
  1. Continuous
  2. Differentiable

Let $\delta L = \nabla_y \delta \underline{v}$ $\qquad \delta D = \mathrm{sym}(\delta L)$

Define virtual work as

$$\delta P = \int_V \left(\sigma : \delta D - \rho \underline{b} \cdot \delta \underline{v} + \rho \frac{\partial v}{\partial t}\Big|_{\underline{x}} \delta \underline{v} \right) dV - \int_S \underline{t} \cdot \delta \underline{v} \, dA$$

Principle of virtual work

1. (Not very useful) if σ satisfies (i) then $\delta P = 0$
2. Useful $\delta P = 0$ for all $\delta \underline{v}$ then σ satisfies (1)

Prove 1 : Start with BLM

$$\int_V \left(\frac{\partial}{\partial y_i}\sigma_{ij} + \rho b_j - \rho \frac{\partial v_j}{\partial t}\Big|_x \right) \delta v_j \, dV = 0 \qquad (4)$$

Now note $\frac{\partial}{\partial y_i}\sigma_{ij}\, \delta v_j = \frac{\partial(\sigma_{ij}\delta v_j)}{\partial y_i} - \underbrace{\sigma_{ij}\frac{\partial \delta v_j}{\partial y_i}}$

Note $\int_V \frac{\partial(\sigma_{ij}\delta v_j)}{\partial y_i}\, dV = \int_S \underbrace{n_i \sigma_{ij}}_{t_j}\, \delta v_j \, dA$ $\quad \sigma_{ij}\delta L_{ji} = \sigma_{ij}\delta D_{ij}$

Subst back in (4) :

$$\int_V \left(-\sigma_{ij}\delta D_{ij} + \rho b_j \delta v_j - \rho \frac{\partial v_j}{\partial t}\Big|_x \delta v_j \right) dV + \int_S t_i \delta v_i \, dA = 0$$

$$\Rightarrow \; -\delta P = 0$$

Version (2)

$$\delta P = \int_V \left(\sigma_{ij}\, \delta D_{ij} - \rho b_j \delta v_j + \rho \frac{\partial v_j}{\partial t}\Big|_{\underline{x}} \delta v_j \right) dV - \int_S t_j \delta v_j \, dA = 0 \quad \forall \delta \underline{v}$$

Note $\sigma_{ij}\, \delta D_{ij} = \sigma_{ij}\, \delta L_{ji} = \sigma_{ij} \frac{\partial \delta v_j}{\partial y_i}$

$$= \frac{\partial}{\partial y_i}\left(\sigma_{ij}\, \delta v_j\right) - \frac{\partial \sigma_{ij}}{\partial y_i} \delta v_j$$

Apply div. theorem to first term, collect terms (b)

$$\int_V \left(\frac{\partial \sigma_{ij}}{\partial y_i} - \rho b_j + \rho \frac{\partial v_j}{\partial t}\Big|_{\underline{x}} \right) \delta v_j \, dV + \int_S \left(n_i \sigma_{ij} - t_j \right) \delta v_j \, dA = 0 \quad \forall \delta \underline{v}$$

(a)

Choose $\delta v_j = f^2(y)\left(\frac{\partial \sigma_{ij}}{\partial y_i} - \rho b_j + \rho \frac{\partial v_j}{\partial t}\Big|_{\underline{x}}\right)$

$f(y) = 0$ on S

(b) $= 0$ (a) integrand is a square – positive or zero everywhere in V

Hence BLM follows (integral can only vanish if integrand $= 0$)

Similarly let $\delta v_j = t_j - \sigma_{ij} n_i$ on S

$\Rightarrow$ integrand in (b) vanishes

$\Rightarrow \quad \sigma_{ij} n_i = t_j$