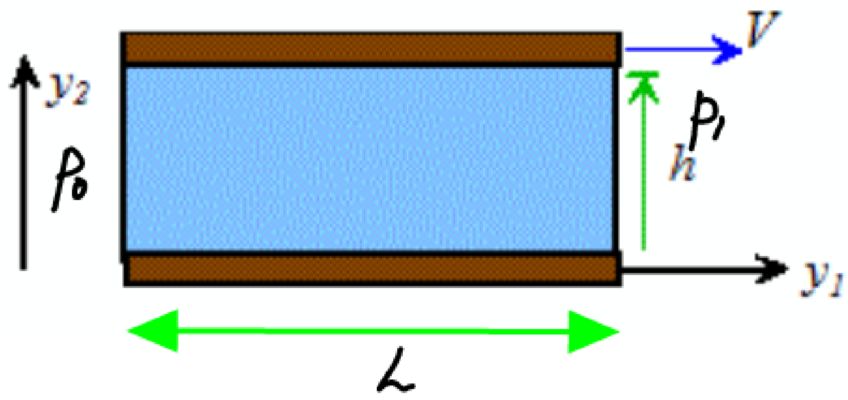
Solutions for Stokes flow

Example 1: Flow between parallel plates

Navier-Stokes

$$-\frac{1}{\rho}\frac{\partial p}{\partial y_i} + \frac{\eta}{\rho}\frac{\partial^2 v_i}{\partial y_j \partial y_j} = 0$$



Assume $\underline{v} = v(y_2)\,\underline{e}_1$

$$\Rightarrow \quad \frac{\partial p}{\partial y_1} = \eta \frac{\partial^2 v}{\partial y_2^2} \qquad \frac{\partial p}{\partial y_2} = \frac{\partial p}{\partial y_3} = 0$$

Hence $p = A y_1 + B \qquad v = \frac{A y_2^2}{2\eta} + C y_2 + D$

Boundary conditions $p = p_0 \;\; y_1 = 0 \qquad p = p_1 \;\; y_1 = L$

$$\Rightarrow A = \frac{p_1 - p_0}{L} \qquad B = p_0$$

$$v = 0 \quad y_2 = 0 \qquad v = V \quad y_2 = h$$

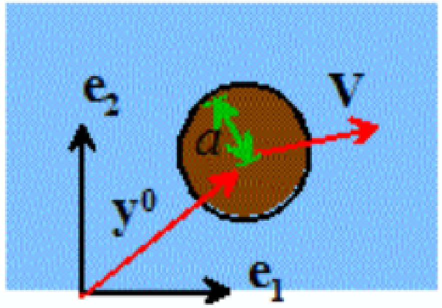
$$\Rightarrow D = 0 \qquad C = \frac{V}{h} - \frac{(p_1 - p_0)h}{2L\eta}$$

Hence $\boxed{v = \frac{V y_2}{h} + \frac{(p_1 - p_0)}{2\eta L}\left(y_2^2 - y_2 h\right)}$

Stress in fluid $\boxed{\sigma = -pI + \eta \frac{\partial v}{\partial y_2}\left(\underline{e}_1 \otimes \underline{e}_2 + \underline{e}_2 \times \underline{e}_1\right)}$

Example 2: Stokes flow around a moving sphere

$$v_i = \frac{3a(a^2-R^2)}{4R^5} V_j (y_j - y_j^0)(y_i - y_i^0) + \frac{a \bar{V}_i}{4R^3}(3R^2 + a^2)$$



Stress state $\sigma_{ij} = -p\,\delta_{ij} + \eta\left(\frac{\partial v_i}{\partial y_j} + \frac{\partial v_j}{\partial y_i}\right)$

Find p from N-S $-\frac{\partial p}{\partial y_i} + \eta \frac{\partial^2 v_i}{\partial y_j \partial y_j} = 0$

$$p = + \frac{3\eta a y_i V_i}{2R^4}$$

Resultant force on sphere $\int_{A_{sphere}} \sigma_{ij} n_j \, dA$

(Use polar coords) $F_i = -6\pi\eta a \bar{V}_i$

Example compressible flow problem : Acoustics

Assumptions :

(1) Air is ideal gas
(2) Irrotational flow
(3) Pressure, density, temperature fluctuations $\delta p, \delta \rho, \delta \theta$ are small ; velocity is small
(4) Neglect heat flux

Solution method

(1) Calculate flow potential Ω satisfying

$$\frac{\partial^2 \Omega}{\partial t^2} = C_s^2 \nabla_y^2 \Omega \qquad C_s = \sqrt{\left.\frac{\partial p}{\partial \rho}\right|_{s=\text{const}}} = \gamma R \theta_0$$

(sound speed) $\qquad \gamma = c_p / c_v$

R - gas const

Ω must satisfy boundary conditions

(a) $\frac{\partial \Omega}{\partial y_i} n_i = \bar{V}_n^*$ on surfaces with known normal velocity V_n^*

(b) $-\rho_0 \frac{\partial \Omega}{\partial t} = p^*$ on surfaces with prescribed pressure

Then velocity and pressure follow from

$$v_i = \frac{\partial \Omega}{\partial y_i} \qquad p = -\rho_0 \frac{\partial \Omega}{\partial t}$$

Derivation:

(1) Irrotational flow $\Rightarrow$ $\nabla_y \times \underline{v} = 0$

Hence $\underline{v} = \nabla_y \Omega$ for some Ω t.b.d.

(2) Navier-Stokes (neglecting quadratic terms in velocity)

$$-\frac{\partial p}{\partial y_i} + \quad \approx \rho \frac{\partial v_i}{\partial t} \qquad (*)$$

Let $p = p_0 + \delta p$, $\rho = \rho_0 + \delta\rho$; $\frac{\partial}{\partial t}$ $(*)$

$$= \frac{\partial}{\partial y_i}\left(\frac{\partial \delta p}{\partial t} + \rho_0 \frac{\partial^2 \Omega}{\partial t^2}\right) = 0$$

Now let $\delta p = \left.\frac{\partial p}{\partial \rho}\right|_{s=\text{const}} \delta\rho = c_s^2\,\delta\rho$

$$\Rightarrow \frac{\partial}{\partial y_i}\left(c_s^2 \frac{\partial \delta\rho}{\partial t} + \rho_0 \frac{\partial^2 \Omega}{\partial t^2}\right) = 0$$

Mass conservation $\Rightarrow \quad \frac{\partial \delta\rho}{\partial t} + \rho_0 \frac{\partial v_i}{\partial y_i} = 0$

$$\Rightarrow \frac{\partial}{\partial y_i}\left(-c_s^2 \frac{\partial^2 \Omega}{\partial y_i \partial y_i} + \frac{\partial^2 \Omega}{\partial t^2}\right) = 0$$

$$\Rightarrow \boxed{c_s^2 \nabla_y^2 \Omega = \frac{\partial^2 \Omega}{\partial t^2}}$$

(can assume const of integration $= 0$)

Finally $\frac{\partial \delta\rho}{\partial t} = \frac{1}{c_s^2}\frac{\partial \delta p}{\partial t} = -\rho_0 \frac{\partial^2 \Omega}{\partial y_i \partial y_i} = -\frac{\rho_0}{c_s^2}\frac{\partial^2 \Omega}{\partial t^2}$

$$\Rightarrow \boxed{\delta p = -\rho_0 \frac{\partial \Omega}{\partial t}}$$

Calculating sound speed

For ideal gas $\psi = c_v\theta - \theta(c_v\log\theta - R\log\rho + s_0)$

Recall $p = \rho^2 \frac{\partial\psi}{\partial\rho}$ $\quad s = -\frac{\partial\psi}{\partial\theta}$

$\Rightarrow \quad s = c_v\log\theta - R\log\rho + s_0$

$\Rightarrow \quad s - s_0 = \log(\theta^{c_v}\rho^{-R})$

$\Rightarrow \quad \theta = \rho^{R/c_v}\exp((s-s_0)/c_v)$

(Recall $\frac{c_p}{c_v} = 1 + \frac{R}{c_v} = \gamma \quad \Rightarrow \quad \frac{R}{c_v} = \gamma - 1$)

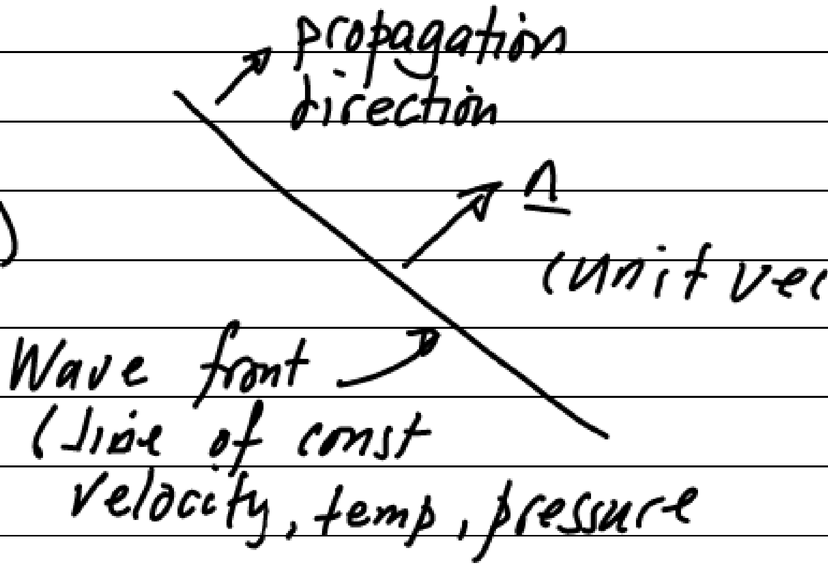
$$p = \rho^2\frac{\partial\psi}{\partial\rho} = \rho R\theta$$

Hence $p = k . R \rho^{\gamma}$ $\qquad k = \exp(s - s_0)/c_v$

$$\left.\frac{\partial p}{\partial \rho}\right|_{s=const} = \gamma k R \rho_0^{\gamma-1} = \gamma R \Theta_0$$

Example Solutions :

(1) Plane Waves



$$\Omega = f(t - y_i n_i / c_s) + g(t + y_i n_i / c_s)$$

$$v_i = -\frac{n_i}{c_s} f'(t - y_i n_i / c_s)$$

$$+ \frac{n_i}{c_s} g'(t + y_i n_i / c_s)$$