School of Engineering
Brown University

EN234: Computational methods in Structural and Solid Mechanics

Homework 5: Advanced elements – small strain B-bar element
Due Wed Oct 21, 2015

In this homework you will implement a small-strain version of a B-bar element that eliminates volumetric locking in near-incompressible materials. This is probably the easiest HW of the entire semester….

The basic idea was discussed in class (see the notes for Lecture 9). You can also find a (somewhat inefficient) MATLAB example on the notes page of the website.

Your goal is to implement a more efficient version of the B-bar method in EN234FEA.

One way to simplify the procedure is to introduce the volume averaged shape function derivatives

$$\overline{\frac{\partial N^a}{\partial x_i}} = \frac{1}{V_{el}} \int_{V_{el}} \frac{\partial N^a}{\partial x_i} dV$$

This can be stored in a two-dimensional matrix `dNbardx(1:n_nodes,1:3)` (for a 3D problem; use 1:2 for a 2D problem) Note that in EN234FEA, the Element_Utilities module pre-defines a set of arrays that are intended to be used for this purpose:

```
    real (prec) :: vol_avg_shape_function_derivatives_1D(3,1)
    real (prec) :: vol_avg_shape_function_derivatives_2D(9,2)
    real (prec) :: vol_avg_shape_function_derivatives_3D(27,3)
```

You can simply 'use' these variables in your code using the same approach that you used to define *N* and *dNdx*.

In matrix form, the element stiffness and residual vectors are

$$\mathbf{r} = \int_{V_e^{(l)}} \overline{\mathbf{B}}^T \boldsymbol{\sigma} dV \qquad \mathbf{k} = \int_{V_e^{(l)}} \overline{\mathbf{B}}^T \mathbf{D} \overline{\mathbf{B}} dV$$

This is just the usual expression, with $\overline{\mathbf{B}}$ instead of $\mathbf{B}$. From this it is clear that implementing the B-bar method requires just a small modification to a standard small-strain solid element.

For a 3D element, the vectors and matrices in this expression are

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} \qquad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1113} & C_{1123} \\ C_{2211} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix}$$

$$
\mathbf{B} = \begin{bmatrix}
\frac{\partial N^1}{\partial y_1} & 0 & 0 & \frac{\partial N^2}{\partial y_1} & \cdots \\
0 & \frac{\partial N^2}{\partial y_2} & 0 & & \frac{\partial N^2}{\partial y_1} \\
0 & 0 & \frac{\partial N^2}{\partial y_3} & & \\
\frac{\partial N^1}{\partial y_2} & \frac{\partial N^1}{\partial y_1} & 0 & \frac{\partial N^2}{\partial y_2} & \frac{\partial N^2}{\partial y_1} \\
\vdots & & & & \\
 & & & & \ddots
\end{bmatrix}
$$

We can incorporate the correction for volumetric locking very simply, by defining

$$
\overline{\mathbf{B}} = \mathbf{B} + \frac{1}{3} \begin{bmatrix}
\left(\overline{\frac{\partial N^1}{\partial y_1}} - \frac{\partial N^1}{\partial y_1} \right) & \left(\overline{\frac{\partial N^1}{\partial y_2}} - \frac{\partial N^1}{\partial y_2} \right) & \left(\overline{\frac{\partial N^1}{\partial y_3}} - \frac{\partial N^1}{\partial y_3} \right) & \left(\overline{\frac{\partial N^2}{\partial y_1}} - \frac{\partial N^2}{\partial y_1} \right) & \cdots \\
\left(\overline{\frac{\partial N^1}{\partial y_1}} - \frac{\partial N^1}{\partial y_1} \right) & \left(\overline{\frac{\partial N^1}{\partial y_2}} - \frac{\partial N^1}{\partial y_2} \right) & \left(\overline{\frac{\partial N^1}{\partial y_3}} - \frac{\partial N^1}{\partial y_3} \right) & & \\
\left(\overline{\frac{\partial N^1}{\partial y_1}} - \frac{\partial N^1}{\partial y_1} \right) & \left(\overline{\frac{\partial N^1}{\partial y_2}} - \frac{\partial N^1}{\partial y_2} \right) & & & \\
0 & 0 & 0 & & \\
 & & & \ddots &
\end{bmatrix}
$$

(The 1/3 becomes a ½ for plane strain, and some rows/columns must be removed).

Update your EN234FEA code to incorporate the B-bar element. You don't need to create a new subroutine for this; you can simply put some conditional statements in your existing code to make it function either as a standard element or a B-bar element, as the user wishes. Behavior can be controlled by the 'element identifier' variable. Note that both the element stiffness and the 'field variables' subroutines should use the same B-bar matrix, for consistency

As usual, hand in a 1 or 2 page description of the tests you ran to check your code, and push your revised code to GitHub. You should modify at least the 3D element; and optionally your 2D small-strain element as well.