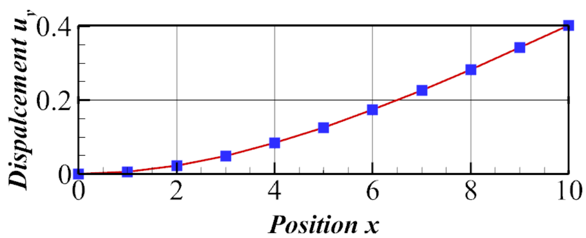
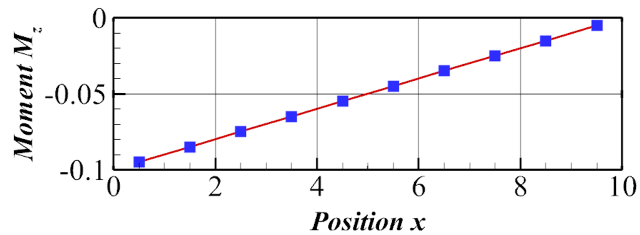
School of Engineering
Brown University

EN234: Computational methods in Structural and Solid Mechanics

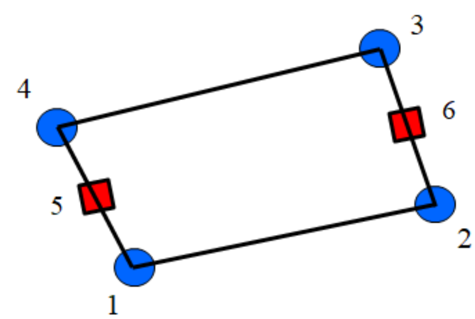
Homework 10: Continuum Beam Elements
Due Fri Dec 1, 2017





In this problem you will code an ABAQUS UEL subroutine to implement the '2D continuum beam' element discussed in class. The basic idea is to modify a standard 2D finite element interpolation scheme in some way so as to describe the variations of stress and strain in a slender beam or thin shell accurately.

Here, we will analyze deformation in a 2D shear-flexible ("Timoshenko") beam by adapting a standard 4 noded plane stress quadrilateral element, as shown in the figure. The beam has thickness h and out-of-plane width w.



The beam element has two 'master' nodes (5,6 in the figure, but of course in your code these are the actual physical two nodes on your element), which each have 3 degrees of freedom (two displacement components, and the rotation of the beam cross section). The degrees of freedom at nodes 5 and 6 are the unknowns in the finite element equations. The element also has a set of internal 'slave' nodes, which have displacement degrees of freedom. Following the standard assumptions of Timoshenko beam theory, we assume that cross-sections normal to the beam centerline remain plane (but may rotate – this is why the element approximates 'Timoshenko' beams rather than the standard Euler-Bernoulli beam). Elementary geometry then gives the coordinates of the slave nodes

$$
\begin{aligned}
x_1^4 &= x_1^5 - (h/2)\sin\theta_1 & x_2^4 &= x_2^5 + (h/2)\cos\theta_1 \\
x_1^1 &= x_1^5 + (h/2)\sin\theta_1 & x_2^1 &= x_2^5 - (h/2)\cos\theta_1 \\
x_1^3 &= x_1^6 - (h/2)\sin\theta_2 & x_2^3 &= x_2^6 + (h/2)\cos\theta_2 \\
x_1^2 &= x_1^6 + (h/2)\sin\theta_2 & x_2^2 &= x_2^6 - (h/2)\cos\theta_2
\end{aligned}
$$

where x_i^a are the displaced positions of the nodes (i.e. after deformation) and

$$\sin\theta_1 = \frac{x_2^6 - x_2^5}{L}\cos\theta^5 + \frac{x_1^6 - x_1^5}{L}\sin\theta^5 \qquad \cos\theta_1 = \frac{x_1^6 - x_1^5}{L}\cos\theta^5 - \frac{x_2^6 - x_2^5}{L}\sin\theta^5$$

$$\sin\theta_2 = \frac{x_2^6 - x_2^5}{L}\cos\theta^6 + \frac{x_1^6 - x_1^5}{L}\sin\theta^6 \qquad \cos\theta_2 = \frac{x_1^6 - x_1^5}{L}\cos\theta^6 - \frac{x_2^6 - x_2^5}{L}\sin\theta^6$$

$$L = \sqrt{(x_1^6 - x_1^5)^2 + (x_2^6 - x_2^5)^2}$$

For small deformations, we can approximate these equations with $\theta^5 \approx 0, \quad \theta^6 \approx 0$.

The infinitesimal displacements of the slave nodes are related to those on the master nodes by

$$\mathbf{v}^s = \mathbf{T}\mathbf{v}^m \qquad \mathbf{v}^m = \begin{bmatrix} v_1^5 & v_2^5 & \omega^5 & v_1^6 & v_2^6 & \omega^6 \end{bmatrix} \qquad \mathbf{v}^s = \begin{bmatrix} v_1^1 & v_2^1 & v_1^2 & \cdots \end{bmatrix}$$

where

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & x_2^5 - x_2^1 & 0 & 0 & 0 \\ 0 & 1 & -(x_1^5 - x_1^1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & x_2^6 - x_2^2 \\ 0 & 0 & 0 & 0 & 1 & x_1^6 - x_1^2 \\ 0 & 0 & 0 & 1 & 0 & x_2^6 - x_2^3 \\ 0 & 0 & 0 & 0 & 1 & x_1^6 - x_1^3 \\ 1 & 0 & x_2^5 - x_2^4 & 0 & 0 & 0 \\ 0 & 1 & -(x_1^5 - x_1^4) & 0 & 0 & 0 \end{bmatrix}$$

The infinitesimal strains in the continuum element can then be calculated using

$$\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{T}\mathbf{u}^m$$

where **B** is the usual matrix mapping the displacements of the 4 noded element to the strain vector.

To calculate the stress in the beam it is helpful to re-write the strains as components in a basis that is oriented parallel and perpendicular to the beam. We can define Laminar basis vectors $\hat{\mathbf{e}}^{(\alpha)}$ oriented with the beam with components

$$\hat{e}_i^{(1)} = \frac{1}{\left|\frac{\partial \mathbf{x}}{\partial \xi_i}\right|}\frac{\partial \mathbf{x}}{\partial \xi_i} \qquad \mathbf{x} = \sum_{a=1}^{4} N^a(x_i)\mathbf{x}^a$$

Strain components in this basis can be calculated as

$$\hat{\boldsymbol{\varepsilon}} = \mathbf{R}\boldsymbol{\varepsilon} \qquad \hat{\boldsymbol{\varepsilon}} = \begin{bmatrix} \hat{\varepsilon}_{11} \\ 2\hat{\varepsilon}_{12} \end{bmatrix} \qquad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} \quad \mathbf{R} = \begin{bmatrix} \left(\hat{e}_1^{(1)}\right)^2 & \left(\hat{e}_2^{(1)}\right)^2 & \hat{e}_1^{(1)}\hat{e}_2^{(1)} \\ 2\hat{e}_1^{(1)}\hat{e}_1^{(2)} & 2\hat{e}_2^{(1)}\hat{e}_2^{(2)} & \hat{e}_1^{(1)}\hat{e}_2^{(2)} + \hat{e}_2^{(1)}\hat{e}_1^{(2)} \end{bmatrix}$$

Slender beam theory assumes that the only non-zero stress components in the beam are $\hat{\sigma}_{11}, \hat{\sigma}_{12}$. You can test your beam element with any material model you like, but it is a good idea to start with an isotropic linear elastic material, for which

$$\hat{\boldsymbol{\sigma}} = \mathbf{D}\hat{\boldsymbol{\varepsilon}} \qquad \mathbf{D} = \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix}$$

Where E is Young's modulus and G is the shear modulus (you can use $G = E/2(1+\nu)$, particularly for short, thick beams but sometimes smaller values of G are used as a penalty coefficient that keeps plane sections that start perpendicular to the centerline of the beam approximately perpendicular to the centerline after deformation.

With these definitions, the element stiffness and residual can be calculated as

$$\mathbf{k}^{el} = wL\frac{h}{2}\int_{-1}^{1}[\mathbf{RBT}]^T\mathbf{D}[\mathbf{RBT}]\,d\xi_2 \qquad \mathbf{r}^{el} = -wL\frac{h}{2}\int_{-1}^{1}[\mathbf{RBT}]^T\boldsymbol{\sigma}\,d\xi_2$$

The integrals should be computed along the line $\xi_1 = 0$ inside the quadrilateral continuum element, using a trapezoidal or Simpson's integration scheme (1D Gauss quadrature works as well if you prefer).

(As an optional experiment, you could try computing the integrals over the quadrilateral elements using the usual 4 point Gauss scheme – you will find that the element locks badly)

For post-processing, store 3 state variables in the element, which store the internal forces in the beam (the axial force; shear force, and bending moment)

$$T = w\frac{h}{2}\int_{-1}^{+1}\sigma_{11}d\xi_2 \qquad V = w\frac{h}{2}\int_{-1}^{+1}\sigma_{12}d\xi_2 \qquad M = w\frac{h^2}{4}\int_{-1}^{+1}\sigma_{11}\xi_2 d\xi_2$$

You can find a sample input file for this problem (named Abaqus_uel_continuum_beam.in) in EN234FEA. The file defines a 10 unit long cantilever beam with 1x1 cross section subjected to a point force at its end. Two output files (beam_displacements.dat and beam_forces.dat) contain Tecplot readable files with the beam displacements and internal forces.

As a check, plot the beam displacements and bending moment predicted by your code and compare the solution to the Euler-Bernoulli result.

Repeat the calculation for a beam with thickness $h=4$. Why does the FEA solution predict larger displacements than Euler_Bernoulli?

As optional additional problems you could try:
(1) Try setting up your code to work with elastic-plastic materials, and devise a test to show that the code works;
(2) Code and test distributed forces for the element

As a solution to this problem
(1) Upload a 1-2 page description of tests that you ran to check your code to Canvas
(2) Push your code to your Github fork.