Review: Hybrid elements for incompressible materials

- Re-derive equations for fully incompressible linear elasticity ab-initio

Field equations: $\sigma_{ij} = S_{ij} + \frac{p}{3}\delta_{ij}$ $\quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ $\quad \frac{1}{3}\varepsilon_{kk} = \frac{1}{3}\frac{\partial u_k}{\partial x_k} = 0$

$$S_{ij} = 2\mu\varepsilon_{ij} \qquad \frac{\partial \sigma_{ij}}{\partial x_i} = 0 \qquad u_i = u_i^* \quad x_i \in S_1 \qquad \sigma_{ij} n_i = t_j^* \quad x_i \in S_2$$

Obtain weak form: $\int_V \left(2\mu \frac{\partial \varepsilon_{ij}}{\partial x_j} + \frac{1}{3}\frac{\partial p}{\partial x_i}\right)\eta_i dV = 0$ $\qquad \int_V \frac{1}{3}\frac{\partial u_k}{\partial x_k} q dV = 0$

Integrate by parts: $\int_V \left(2\mu\varepsilon_{ij}\frac{\partial \eta_i}{\partial x_j} + \frac{p}{3}\frac{\partial \eta_i}{\partial x_i}\right)dV - \int_V b_i \eta_i dV - \int_{S_2} t_i^* \eta_i dA = 0$ $\qquad \int_V \frac{1}{3}\frac{\partial u_k}{\partial x_k} q dV = 0$

FE interpolation $\quad u_i = N^a(\mathbf{x}) u_i^a \qquad \eta_i = N^a(\mathbf{x})\eta_i^a$

$$p = M^a(\mathbf{x}) p^a \qquad q = M^a(\mathbf{x}) q^a$$

$M^a(\mathbf{x})$ are interpolation functions for pressure
Must be one order lower than interpolation for displacement
Need not be continuous across element boundaries

Insert intepolation into weak form to get FE eqs

Define $C^{DEV}_{ijkl} = \mu\left(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}\right)$

$$\left\{ \int_R \left(\underbrace{C^{DEV}_{ijkl} \frac{\partial N^b}{\partial x_l} \frac{\partial N^a}{\partial x_j}}_{K^{uu}_{aibk}} u_k^b + \underbrace{\frac{1}{3} \frac{\partial N^a}{\partial x_i} M_b}_{K^{up}_{aib}} p^b \right) dV - \int t_i^* N^a \right\} \eta_i^a = 0$$

$\forall$ admiss η_i^a

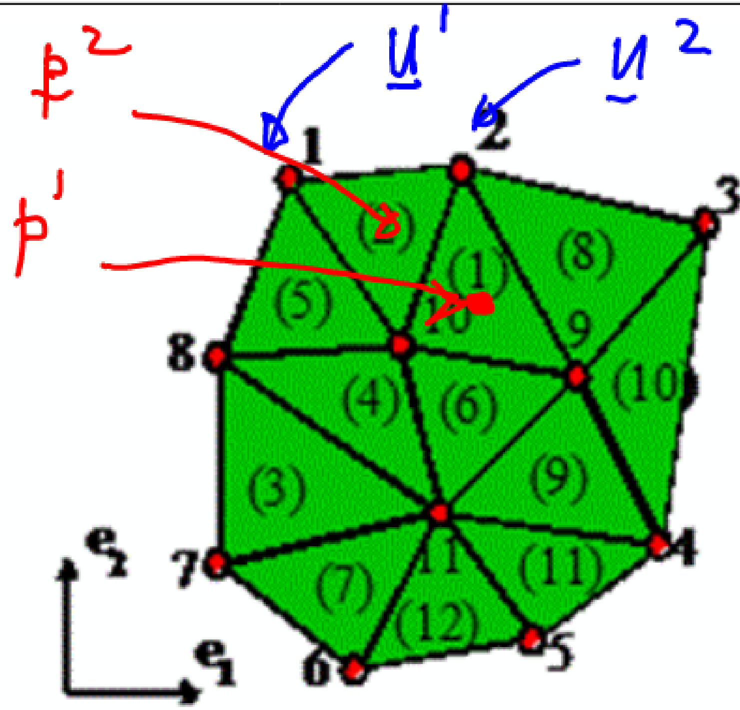
$$\left\{ \int_R \underbrace{\frac{1}{3} \frac{\partial N^b}{\partial x_i} M^a}_{K^{pu}_{abi}} u_i^b \right\} q^a = 0 \quad \forall \text{ admiss } q^b$$

FE equations follow

$$K^{uu}_{aibk} u_k^b + K^{up}_{aib} p^b = f_i^a$$

$$K^{pu}_{abi} u_i^b = 0$$

Implementing hybrid elements



Global unknowns $\{\underline{u}, p\}$

$\underline{u}$ - usual vector of displacements

Linear elements: piecewise constant interpolation for pressure
$\Rightarrow$ 1 unknown pressure DOF for each element

Quadratic: use linear interpolation for pressure
- 3 unknown pressures @ element corners for triangles
- 4 " " " " quads
- 8 " " " " bricks

Pressures need not be continuous @ element boundaries

Assembly is done element-by element as before

$$[K]\begin{bmatrix}\underline{u}\\ \underline{p}\end{bmatrix} = \begin{bmatrix}\underline{f}\\ \underline{0}\end{bmatrix}$$

Focus on element level operations

$$[k^{el}] = \begin{bmatrix}K^{uu} & K^{up}\\ K^{pu} & 0\end{bmatrix}\begin{bmatrix}\underline{u}\\ \underline{p}\end{bmatrix}$$

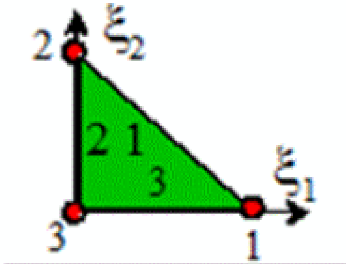
$$[k^{uu}] = \int_{\Omega_{el}} [B]^T [D^{Dev}] [B]\, dV$$

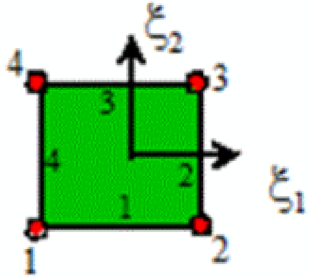
$$[k^{up}] \quad \int_{\Omega_{el}} \frac{1}{3}[B^T]\,[M]\, dV$$

Compute with full Gauss quadrature

$[k^{pu}] = [k^{up}]^T$ $\qquad$ $[M]$ maps $\underline{p}$ onto $\underline{\sigma}$: $\underline{\sigma} = [M]\,\underline{p}$

Table of [M]



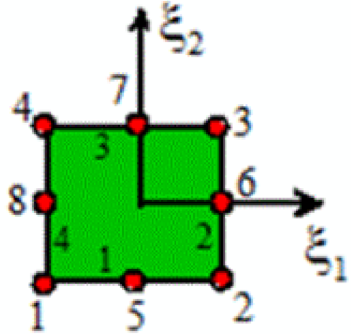


$$[M] = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

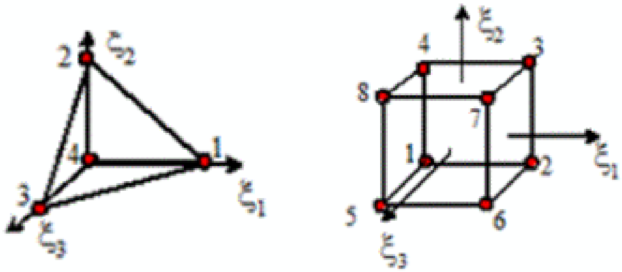
$[D^{DEV}] =$ (3D version)

$$\mu \begin{bmatrix} 2 & & & & & \\ & 2 & & & 0 & \\ & & 2 & & & \\ & & & 1 & & \\ & 0 & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

$$N^1 = 0.25(1-\xi_1)(1-\xi_2)$$
$$N^2 = 0.25(1+\xi_1)(1-\xi_2)$$
$$N^3 = 0.25(1+\xi_1)(1+\xi_2)$$
$$N^4 = 0.25(1-\xi_1)(1+\xi_2)$$



$$[M] = \begin{bmatrix} N^1 & N^2 & N^3 & N^4 \\ N^1 & N^2 & N^3 & N^4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



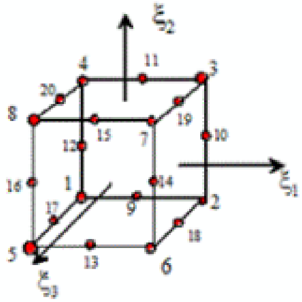
$$[M] = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$N^1 = (1-\xi_1)(1-\xi_2)(1-\xi_3)/8 \quad N^2 = (1+\xi_1)(1-\xi_2)(1-\xi_3)/8$

$N^3 = (1+\xi_1)(1+\xi_2)(1-\xi_3)/8 \quad N^4 = (1-\xi_1)(1+\xi_2)(1-\xi_3)/8$

$N^5 = (1-\xi_1)(1-\xi_2)(1+\xi_3)/8 \quad N^6 = (1+\xi_1)(1-\xi_2)(1+\xi_3)/8$

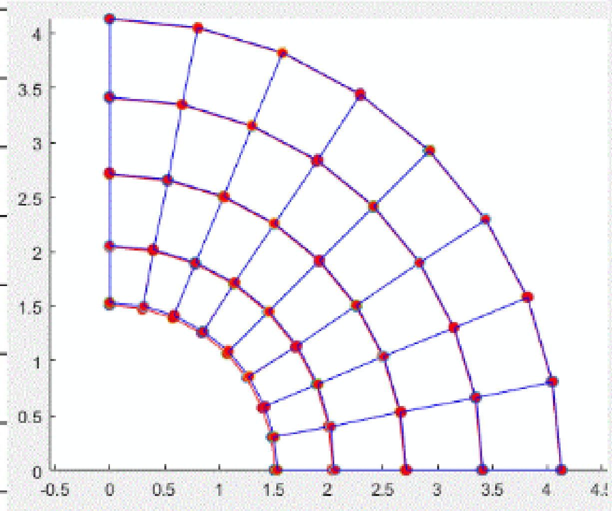
$N^7 = (1+\xi_1)(1+\xi_2)(1+\xi_3)/8 \quad N^8 = (1-\xi_1)(1+\xi_2)(1+\xi_3)/8$



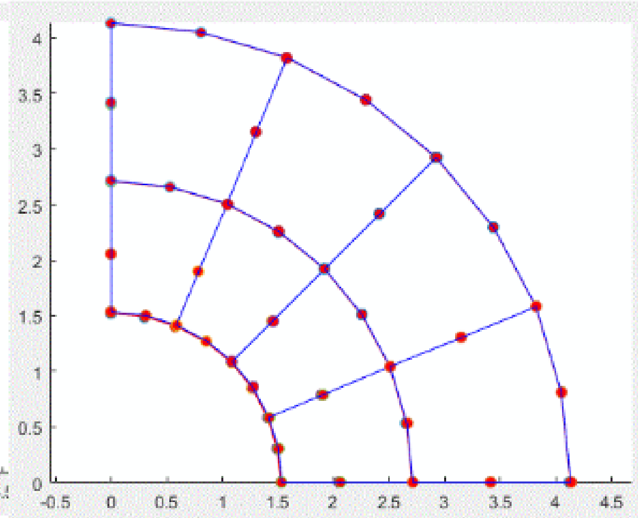
$$[M] = \begin{bmatrix} N^1 & N^2 & N^3 & \\ N^1 & N^2 & N^3 & \\ N^1 & N^2 & N^3 & \cdots \\ 0 & & & \\ 0 & & & \\ 0 & & & \end{bmatrix}$$

Test:

Linear



Quadratic



Advantages: Works for fully incompressible materials

Disadvantages: Can't handle compressible solids
Solve for more global DOF
Zeros on diagonal can be a problem for some solvers

6.6 Augmented Lagrangean hybrid elements

Goal: Develop hybrid scheme for compressible materials

Approach: Solve separately for pressure-volume & deviatoric relations

$$\sigma_{ij} = S_{ij} + p\,\delta_{ij} \qquad p = \sigma_{kk}/3$$

$$S_{ij} = C^{DEV}_{ijkl}\,\varepsilon_{kl}$$

$$\boxed{p = K\,\varepsilon_{kk} = K\frac{\partial u_k}{\partial x_k}}$$

New equation

$$K = \text{Bulk modulus} = \frac{E}{3(1-2\nu)}$$

$$C_{ijkl} = \frac{E}{2(1+\nu)}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \frac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl}$$

$$\Rightarrow C^{DEV}_{ijkl} = C_{ijkl} - \frac{E}{3(1-2\nu)}\delta_{ij}\delta_{kl} = \frac{E}{2(1+\nu)}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) - \frac{E}{3(1+\nu)}\delta_{ij}\delta_{kl}$$

Interpolated weak form of governing equations

$$\left\{ \int_R \left(\underbrace{C^{DEV}_{ijkl} \frac{\partial N^b}{\partial x_l} \frac{\partial N^a}{\partial x_j}}_{K^{uu}} u_k^b + \underbrace{\frac{\partial N^a}{\partial x_i} M^b}_{K^{up}} p^b \right) dV - \int t_i^* N^a \right\} \eta_i^a = 0 \quad \forall \eta_i^a$$

Also enforce $p = K \frac{\partial u_k}{\partial x_k}$ in weak form $\left(\frac{\partial u_k}{\partial x_k} - \frac{p}{K} \right) = 0$

$$\left\{ \int_R \underbrace{\frac{\partial N^b}{\partial x_i}}_{K^{pu}} u_i^b M^a - \underbrace{\frac{M^a M^b}{K}}_{K^{pp}} p^b \right\} q^a = 0$$

Assembly of global system is the same as for hybrid element

Focus on element computations

$$[k^{el}] = \begin{bmatrix} k^{uu} & k^{up} \\ k^{pu} & k^{pp} \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{p} \end{bmatrix}$$

$$\left.\begin{aligned} [k^{uu}] &= \int_{\Omega_{el}} [B]^T [D^{Dev}] [B] \, dV \\ [k^{up}] &= [k^{pu}]^T = \int_{\Omega_{el}} [B]^T [m] \, dV \\ [k^{pp}] &= \int_{\Omega_{el}} -\frac{1}{K} \, \underline{m} \otimes \underline{m} \, dV \end{aligned}\right\} \text{Compute with full integration}$$

$\underset{\sim}{m} = [1]$ for linear element

$= [M^1 \; m^2 \; m^3 \; m^4]$ for quadratic quad

This formulation works for all materials

Note that as long as $\kappa \neq \infty$ we can eliminate p inside each element

$$K^{pu}\,\underline{u} + K^{pp}\,p = \underline{0}$$

$$\Rightarrow \quad p = -\,K^{pp^{-1}}\,K^{pu}\,\underline{u}$$

$$K^{uu}\,\underline{u} + K^{up}\,p = \underline{f}$$

$$\Rightarrow \quad \underbrace{\left[K^{uu} - K^{up}\,K^{pp^{-1}}\,K^{pu}\right]}_{\text{Reduced } [K^{el}]}\,\underline{u} = \underline{f}$$

In practice p is usually left in global system

This version is used as hybrid element in ABAQUS