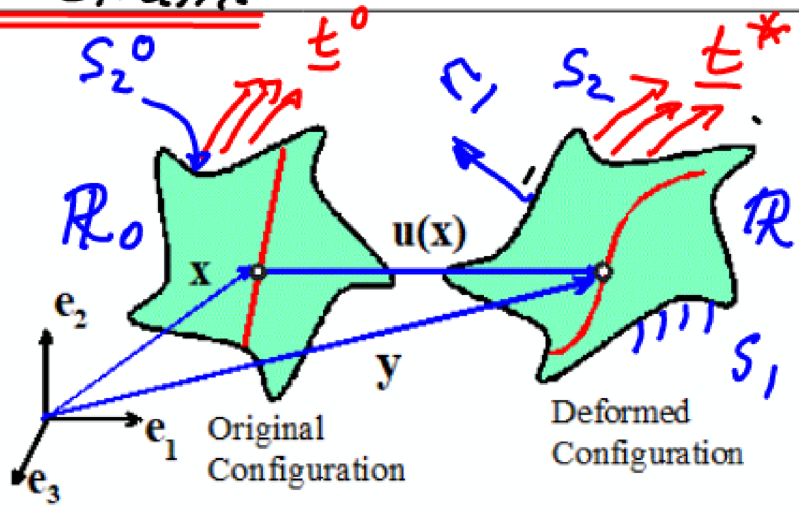
9.2 Hyperelasticity – FEA for finite strains



Governing equations

Def gradient $\dfrac{\partial y_i}{\partial x_j} = \delta_{ij} + \dfrac{\partial u_i}{\partial x_j} = F_{ij}$

- Jocobian $J = \det(F)$
- Cauchy – Green tensors

$$B_{ij} = F_{ik} F_{jk} \qquad C_{ij} = F_{ki} F_{kj}$$

- Stresses : σ_{ij} - Cauchy "true" stress

τ_{ij} - Kirchhoff stress $\tau_{ij} = J\,\sigma_{ij}$

$S_{ij} = F^{-1}_{ik} J \sigma_{kj}$ - First Piola-Kirchhoff / nominal

$\Sigma_{ij} = F^{-1}_{ik} J \sigma_{kl} F^{-1}_{jl}$ 2nd P-K "material"

- Equilibrium $\frac{\partial \sigma_{ji}}{\partial y_j} = 0$ $\frac{\partial S_{ji}}{\partial x_j} = 0$
- Boundary conditions $n_i \sigma_{ij} = t_j^*$ on S_2

  $u_i = u_i^*$ on S_1
- Stress-Strain relation (Neo-Hookean solid)

$$\sigma_{ij} = \frac{1}{J^{5/3}} \mu \left\{ B_{ij} - B_{kk} \frac{\delta_{ij}}{3} \right\} + k(J-1)\delta_{ij}$$

μ - shear modulus $K \gg \mu$ Bulk modulus

(Simple rubber model)

Weak form of equilibrium (Use 1st version)

$$\int_{R} \frac{\partial \sigma_{ji}}{\partial y_j}\, \eta_i \, dV = 0 \qquad \forall \text{ admiss } \eta_i$$

$$\Rightarrow \int_{R} \frac{\partial}{\partial y_i}\left(\sigma_{ji}\eta_i\right) - \sigma_{ji}\frac{\partial \eta_i}{\partial y_j} \Big\}\, dV = 0$$

$$\Rightarrow \int_{R} \sigma_{ji}\frac{\partial \eta_i}{\partial y_j}\, dV - \int_{S_2} t_i^{*}\, \eta_i \, dA = 0 \quad \forall\, \eta_i$$

Map back to ref config

$$\int_{R_0} \underbrace{J\sigma_{ji}}_{=\tau_{ji}} \frac{\partial \eta_i}{\partial y_j}\, dV_0 - \int_{S_2^0} t_i^0\, \eta_i \, dA_0 = 0 \ \forall\, \eta_i$$

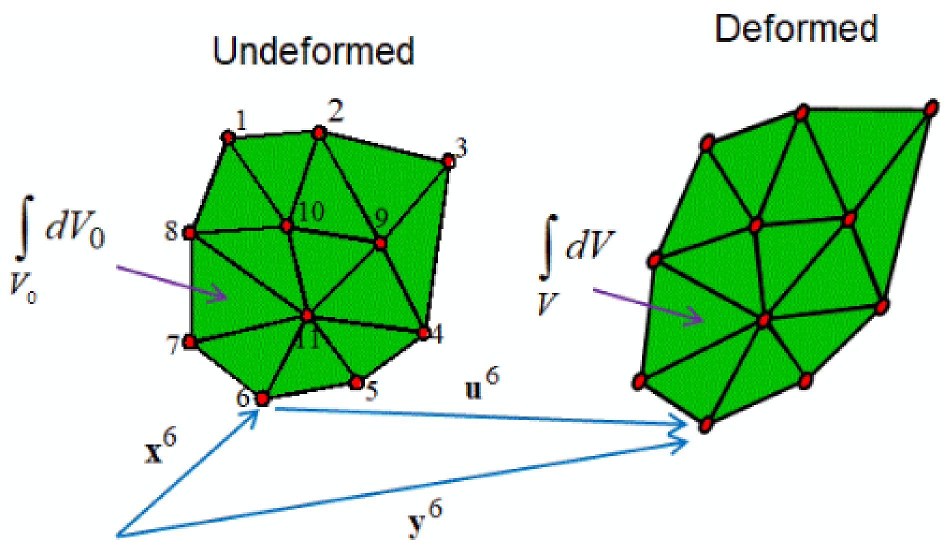
FE approximation $\quad u_i = N^a(\underline{x})\, u_i^a \qquad \eta_i = N^a \eta_i^a$

Discrete equilibrium eq

$$R_i^a = f_i^a$$

$$R_i^a = \int_{R_0} \tau_{ji} \frac{\partial N^a}{\partial y_j}\, dV_0$$

$$f_i^a = \int_{S_2^0} t_i^0 N^a \, dA_0$$



Computing R_i^a

(1) Find $\partial N^a / \partial \xi_i$ & Find $\partial x_i / \partial \xi_j = \sum_a \frac{\partial N^a}{\partial \xi_j} x_i^a$

(2) $\partial N^a / \partial x_j = [\partial N^a / \partial \xi_k][\partial \xi_k / \partial x_j]$

(3) $\frac{\partial N^a}{\partial y_j} = \frac{\partial N^a}{\partial x_k} \frac{\partial x_k}{\partial y_j} = \frac{\partial N^a}{\partial x_k} F_{kj}^{-1}$

Coding : (1) Find $\partial N^a / \partial x_j$ (steps 1, 2)

(2) $F_{ij} = \dfrac{\partial N^a}{\partial x_j} u_i^a + \delta_{ij}$

(3) F_{ij}^{-1}

(4) $\dfrac{\partial N^a}{\partial y_j} = \dfrac{\partial N^a}{\partial x_k} F_{kj}^{-1}$

Express R_i^a as vector (usual convention)

$$\underline{R} = [\, R_1^1 \; R_2^1 \; R_3^1 \; R_1^2 \; \ldots \,]$$

$$\underline{R}^{el} = \int_{\Omega_0} [B]^T \underline{\tau} \, dV_0 \qquad \underline{\tau} = [\, \tau_{11} \; \tau_{22} \; \tau_{33} \; \tau_{12} \ldots \,]$$

$[B]$ is usual $[B]$ but $\partial N^a / \partial x_j$ is replaced by $\partial N^a / \partial y_j$

For example 3D $[B]$ matrix has form shown

$$[B] = \begin{bmatrix} \frac{\partial N^1}{\partial y_1} & 0 & 0 & \frac{\partial N^2}{\partial y_1} & 0 & 0 & \\ 0 & \frac{\partial N^1}{\partial y_2} & 0 & 0 & \frac{\partial N^2}{\partial y_2} & 0 & \\ 0 & 0 & \frac{\partial N^1}{\partial y_3} & 0 & 0 & \frac{\partial N^2}{\partial y_3} & \\ \frac{\partial N^1}{\partial y_2} & \frac{\partial N^1}{\partial y_1} & 0 & \frac{\partial N^2}{\partial y_2} & \frac{\partial N^2}{\partial y_2} & 0 & \cdots\cdots \\ \frac{\partial N^1}{\partial y_3} & 0 & \frac{\partial N^1}{\partial y_1} & & & & \\ 0 & \frac{\partial N^1}{\partial y_3} & \frac{\partial N^1}{\partial y_2} & & & & \end{bmatrix}$$

Vol integrals are computed using same Gauss quadrature we used for small strains

$$\underline{R}^{el} = \sum_{\substack{\text{int pts} \\ i}} [B]^T \underline{\tau} \, W_i$$

We now have nonlinear eqs $R_i^a [U_k^b] = f_i^a$

Solve nonlinear eqs with Newton-Raphson

- Guess $\underline{u} = \underline{w}$ (eg $\underline{w} = 0$ or sol from previous step)
- Correct $\underline{w}$: $[K]d\underline{w} = \underline{f} - \underline{R}[\underline{w}]$

$$[K] = \frac{\partial \underline{R}}{\partial \underline{u}} \quad \text{(consistent tangent)}$$

$$\underline{w} \to \underline{w} + d\underline{w}$$

- Check convergence, repeat as needed

We need $\dfrac{d\underline{R}}{\partial \underline{u}} \equiv \dfrac{\partial R_i^a}{\partial u_k^b} = \dfrac{\partial}{\partial u_k^b}\left\{\int_{B_0} \dfrac{\partial N^a}{\partial y_j} \tau_{ji}\, dV_0\right\}$

$$\equiv K_{aibk}$$

Chain rule

$$K_{aibk} = \underbrace{\int_{R_0} \frac{\partial N^a}{\partial y_i} \frac{\partial \tau_{ji}}{\partial B_{pq}} \frac{\partial B_{pq}}{\partial F_{rs}} \frac{\partial F_{rs}}{\partial u_k^b} \, dV_0}_{\text{Similar to small strains}} + \underbrace{\int_{R} \frac{\partial}{\partial u_k^b} \left\{ \frac{\partial N^a}{\partial y_j} \right\} \tau_{ji} \, dV_0}_{\text{Geometric Stiffness}}$$

Derivatives $\quad F_{rs} = \delta_{rs} + \frac{\partial N^c}{\partial x_s} u_r^c$

$$\Rightarrow \quad \frac{\partial F_{rs}}{\partial u_k^b} = \frac{\partial N^b}{\partial x_s} \delta_{kr} = \frac{\partial N^b}{\partial y_l} F_{ls} \delta_{kr}$$

$$B_{pq} = F_{pn} F_{qn} \Rightarrow \frac{\partial B_{pq}}{\partial F_{rs}} = \delta_{pr} \delta_{ns} F_{qn} + F_{pn} \delta_{qr} \delta_{ns}$$

$$= \delta_{pr} F_{qs} + F_{ps} \delta_{qr}$$

$$\frac{\partial}{\partial u_k^b}\left\{\frac{\partial N^a}{\partial y_j}\right\} = \left\{\frac{\partial N^a}{\partial x_s} F_{sj}^{-1}\right\} \qquad (1)$$

Note $\quad F F^{-1} = I \Rightarrow \frac{\partial F}{\partial \lambda} F^{-1} + F \frac{\partial F^{-1}}{\partial \lambda} = 0$

$$\Rightarrow \frac{\partial F^{-1}}{\partial \lambda} = -F^{-1} \frac{\partial F}{\partial \lambda} F^{-1}$$

$$\Rightarrow (1) = -\frac{\partial N^a}{\partial x_s} F_{sp}^{-1} \frac{\partial F_{pr}}{\partial u_k^b} F_{rj}^{-1}$$

$$= -\underbrace{\frac{\partial N^a}{\partial x_s} F_{sp}^{-1}}_{\frac{\partial N^a}{\partial y_p}} \frac{\partial N^b}{\partial y_l} \overbrace{F_{lr}\, \delta_{kp}\, F_{rj}^{-1}}^{\delta_{jl}} = -\frac{\partial N^a}{\partial y_k} \frac{\partial N^b}{\partial y_j}$$

Combine terms so far

$$K^{el}_{aibk} = \int_{V_0} \frac{\partial N^a}{\partial y_j} \frac{\partial \tau_{ji}}{\partial B_{pq}} \left(\delta_{pk} B_{ql} + \delta_{qk} B_{pl} \right) \frac{\partial N^b}{\partial y_l} dV_0 - \int_{V_0} \frac{\partial N^a}{\partial y_k} \tau_{ji} [F_{kl}] \frac{\partial N^b}{\partial y_j} dV_0$$

Finally we need material tangent stiffness $\frac{\partial \tau_{ji}}{\partial B_{pq}}$

$$\tau_{ji} = \frac{\mu}{J^{2/3}} \left\{ B_{ij} - B_{kk} \frac{\delta_{ij}}{3} \right\} + K J (J-1) \delta_{ij}$$

Note $J = \sqrt{\det(B)}$ $\qquad \frac{\partial \det(A)}{\partial A} = \det(A) A^{-T}$

$$\frac{\partial \tau_{ji}}{\partial B_{pq}} = \frac{\mu}{J^{2/3}} \left\{ \frac{\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}}{2} - \frac{\delta_{pq}\,\delta_{ij}}{3} \right\}$$

Symmetrize to preserve symmetry in i,j & p,q

$$- \frac{\mu}{J^{2/3}} \left\{ B_{ij} - \frac{B_{kk}\,\delta_{ij}}{3} \right\} B^{-1}_{pq}$$

$$+ \; K J \left(J - \frac{1}{2} \right) \delta_{ij} \; B^{-1}_{pq}$$

See notes for L14 for matrix forms for [K] and sample UEL code