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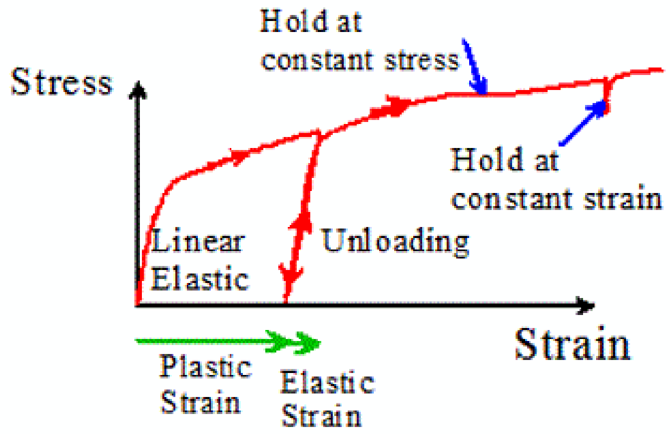
FEA for viscoplastic materials

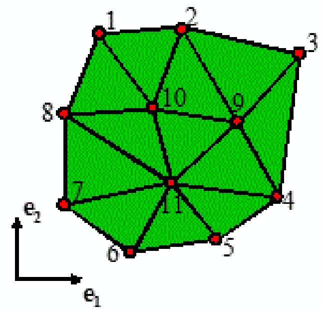
- Material model $\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^{e} + \dot{\varepsilon}_{ij}^{p}$



$$S_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \qquad \sigma_e = \sqrt{\frac{3}{2} S_{ij} S_{ij}}$$

$$\dot{\varepsilon}_{ij}^{e} = \frac{(1+\nu)}{E}\dot{S}_{ij} + \frac{1-2\nu}{3E}\dot{\sigma}_{kk}\delta_{ij} \qquad \dot{\varepsilon}_{ij}^{p} = \dot{\varepsilon}_0 \left(\frac{\sigma_e}{\sigma_0(\varepsilon_e)} \right)^m \frac{3}{2}\frac{S_{ij}}{\sigma_e}$$

$$\sigma_0(\varepsilon_e) = Y\left(1 + \frac{\varepsilon_e}{\varepsilon_0}\right)^{1/n} \qquad \frac{d\varepsilon_e}{dt} = \sqrt{\frac{2}{3}\dot{\varepsilon}_{ij}^{p}\dot{\varepsilon}_{ij}^{p}} = \dot{\varepsilon}_0 \left(\frac{\sigma_e}{\sigma_0}\right)^m$$



Material model is history and strain rate dependent – FEA must do a time integral (time marching) $\{\Delta\varepsilon_{ij}, \sigma_{ij}^{n}, \varepsilon_e^{n}, \Delta t\}$

Assume that $u_i = \sigma_{ij} = \varepsilon_e = 0$ at time *t=0*

Apply loads in a series of steps t_1, t_2, t_3

Satisfy equilibrium (through PVW) at end of each step $\int_R \sigma_{ij}^{n+1}\{\Delta\varepsilon_{ij}, \sigma_{ij}^{n}, \varepsilon_e^{n}, \Delta t\}\frac{\partial \eta_i}{\partial x_j} dV_0 - \int_R b_i \eta_i dV_0 - \int_{\partial_2 R} t_i \eta_i dA = 0$

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FEA Implementation

Introduce usual interpolation $u_i = N^a u_i^a \quad \eta_i = N^a \eta_i^a$

$\Rightarrow$ FEA equations $R_i^a(\Delta u_i, \Delta t) = f_i^a$

Satisfy @ end of each time step - Use Newton-Raphson

Guess $\Delta u_i^a = w_i^a$

Correct $K_{aibk}\, dw_k^b = f_i^a - R_i^a$

$$R_i^a = \int_R \sigma_{ij}^{n+1} (\dots) \frac{\partial N^a}{\partial x_j} dV \qquad f_i^a = \int_{S_2} t_i^* N^a dA$$

$$K_{aibk} = \int_R \frac{\partial \sigma_{ij}^{n+1}}{\partial \Delta \varepsilon_{kl}} \frac{\partial N^a}{\partial x_j} \frac{\partial N^b}{\partial x_l} dV$$

We can implement in ABAQUS UMAT

We need to find σ^{n+1}, ε_e^{n+1}

Given σ_{ij}^{n}, ε_e^{n}, Δt, $\Delta \varepsilon_{ij}$

Compute σ_{ij}^{n+1}, ε_e^{n+1} (or $\Delta \varepsilon_e = \varepsilon_e^{n+1} - \varepsilon_e^{n}$)

and $\dfrac{\partial \sigma_{ij}^{n+1}}{\partial \Delta \varepsilon_{kl}}$

User must code formula for σ_{ij}^{n+1}, DDSDDE

Viscoplastic Stress Update Algorithm

(For a generic integration point)

Given $\{\Delta\varepsilon_{ij}, \sigma_{ij}^{n}, \varepsilon_{e}^{n}, \Delta t\}$ Compute $\{\sigma_{ij}^{n+1}, \varepsilon_{e}^{n+1}\}$

1. Compute $\Delta e_{ij} = \Delta\varepsilon_{ij} - \Delta\varepsilon_{kk}\delta_{ij}/3$ $\quad S_{ij}^{n} = \sigma_{ij}^{n} - \sigma_{kk}^{n}\delta_{ij}/3$

2. Compute $S_{ij}^{*} = S_{ij}^{n} + \dfrac{E}{(1+\nu)}\Delta e_{ij}$ $\quad \sigma_{e}^{*} = \sqrt{3S_{ij}^{*}S_{ij}^{*}/2}$

3. Using Newton-Raphson iteration, solve for $\Delta\varepsilon_{e}$

$$\sigma_{e}^{*} - Y\left(1 + \frac{\varepsilon_{e}^{n} + \Delta\varepsilon_{e}}{\varepsilon_{0}}\right)^{1/n}\left(\frac{\Delta\varepsilon_{e}}{\Delta t\dot{\varepsilon}_{0}}\right)^{1/m} - \frac{3E}{2(1+\nu)}\Delta\varepsilon_{e} = 0$$

4. Calculate $\sigma_{ij}^{n+1} = \left(1 - \dfrac{3E\Delta\varepsilon_{e}}{2(1+\nu)\sigma_{e}^{*}}\right)S_{ij}^{*} + \left(\sigma_{kk}^{n} + \dfrac{E\Delta\varepsilon_{kk}}{(1-2\nu)}\right)\dfrac{\delta_{ij}}{3}$

5. Calculate $\gamma = \dfrac{3E}{2(1+\nu)\sigma_{e}^{*}} + \left(1 - \dfrac{3E\Delta\varepsilon_{e}}{2(1+\nu)\sigma_{e}^{*}}\right)\left(\dfrac{1}{n(\varepsilon_{0} + \varepsilon_{e} + \Delta\varepsilon_{e})} + \dfrac{1}{m\Delta\varepsilon_{e}}\right)$

6. Tangent: $\dfrac{\partial\sigma_{ij}}{\partial\Delta\varepsilon_{kl}} = C_{ijkl}^{ep} = \dfrac{E}{1+\nu}\left(1 - \dfrac{3E\Delta\varepsilon_{e}}{2(1+\nu)\sigma_{e}^{*}}\right)\left(\dfrac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}\right) - \dfrac{1}{3}\delta_{ij}\delta_{kl}\right) + \dfrac{E}{(1+\nu)}\dfrac{9E\left(\Delta\varepsilon_{e} - 1/\gamma\right)}{4(1+\nu)\sigma_{e}^{*}}\dfrac{S_{ij}^{*}}{\sigma_{e}^{*}}\dfrac{S_{kl}^{*}}{\sigma_{e}^{*}} + \dfrac{E}{3(1-2\nu)}\delta_{ij}\delta_{kl}$

Derivations : Recall $\Delta\varepsilon_{ij} = \Delta\varepsilon_{ij}^{e} + \Delta\varepsilon_{ij}^{p}$

Recall $\Delta\varepsilon_{kk}^{p} = 0 \Rightarrow \Delta\varepsilon_{kk} = \Delta\varepsilon_{kk}^{e}$

Hence $\sigma_{kk}^{n+1} = \sigma_{kk}^{n} + \frac{E}{1-2\nu}\Delta\varepsilon_{kk}$ (Elasticity)

4. Calculate $\sigma_{ij}^{n+1} = \left(1 - \frac{3E\Delta\varepsilon_e}{2(1+\nu)\sigma_e^*}\right)S_{ij}^* + \boxed{\left(\sigma_{kk}^{n} + \frac{E\Delta\varepsilon_{kk}}{(1-2\nu)}\right)\frac{\delta_{ij}}{3}}$

Now consider deviatoric stress

Let $\Delta e_{ij} = \Delta\varepsilon_{ij} - \Delta\varepsilon_{kk}\frac{\delta_{ij}}{3}$

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Hence $S_{ij}^{n+1} = S_{ij}^{n} + \frac{E}{1+\nu}\Delta e_{ij}^{e} = \underbrace{S_{ij}^{n} + \frac{E}{1+\nu}}_{S_{ij}^{*}\text{ – elastic predictor}}\left(\Delta e_{ij} - \Delta e_{ij}^{p}\right)$

We need to find Δe_{ij}^{p}

Governing eq $\frac{de_{ij}^{p}}{dt} = \dot{\varepsilon}_0 \left(\frac{\sigma_e}{\sigma_0}\right)^m \frac{3}{2}\frac{S_{ij}}{\sigma_e}$

Also $\frac{d\varepsilon_e}{dt} = \dot{\varepsilon}_0 \left(\frac{\sigma_e}{\sigma_0}\right)^m$

We need to integrate from $t_n < t < t_n + \Delta t$

We assume Δt is small

Assume σ_e, S_{ij} vary slowly - assume $\frac{de_{ij}^{p}}{dt}$ is const

We could estimate de_{ij}/dt using

(1) S_{ij} @ time t_n → "Explicit"

(2) S_{ij} " " $t_n + \Delta t$ → "Implicit"

(3) Use eg midpoint rule

In practice we nearly always use (2)

(2) is stable for large timesteps
- gives symmetric tangents

We will use method (2)

Set $\Delta e_{ij}^{p} = \Delta \varepsilon_e \frac{3}{2} \frac{S_{ij}^{n+1}}{\sigma_e^{n+1}}$ $\qquad \Delta \varepsilon_e = \Delta t \, \dot{\varepsilon}_0 \left(\frac{\sigma_e^{n+1}}{\sigma_0^{n+1}} \right)^{m}$

$$\left. \begin{aligned} &\text{Hence } * \Rightarrow \quad S_{ij}^{n+1} \left(1 + \frac{3E}{2(1+\nu)} \frac{\Delta \varepsilon_e}{\sigma_e^{n+1}} \right) = S_{ij}^{*} \\ &\sqrt{\frac{3}{2} (\,)(\,)} \Rightarrow \quad \sigma_e^{n+1} + \frac{3E}{2(1+\nu)} \Delta \varepsilon_e = \sigma_e^{*} \end{aligned} \right\} **$$

$$\Rightarrow \sigma_e^{*} - \frac{3E}{2(1+\nu)} \Delta \varepsilon_e - \underbrace{\left(\frac{\Delta \varepsilon_e}{\dot{\varepsilon}_0 \Delta t} \right)^{1/m} Y \left(1 + \frac{\varepsilon_e + \Delta \varepsilon_e}{\varepsilon_0} \right)^{1/n}}_{\sigma_e^{n+1}} = 0$$

3. Using Newton-Raphson iteration, solve for $\Delta\varepsilon_e$

$$\sigma_e^* - Y\left(1+\frac{\varepsilon_e^n + \Delta\varepsilon_e}{\varepsilon_0}\right)^{1/n}\left(\frac{\Delta\varepsilon_e}{\Delta t\dot{\varepsilon}_0}\right)^{1/m} - \frac{3E}{2(1+\nu)}\Delta\varepsilon_e = 0$$

Finally to find S_{ij}^{n+1} note ** $\Rightarrow \dfrac{S_{ij}^{n+1}}{\sigma_e^{n+1}} = \dfrac{S_{ij}^*}{\sigma_e^*}$

$$\Rightarrow S_{ij}^{n+1} = \left(1 - \frac{3E\Delta\varepsilon_e}{2(1+\nu)\sigma_e^*}\right) S_{ij}^*$$

Collect terms

4. Calculate

$$\sigma_{ij}^{n+1} = \boxed{\left(1-\frac{3E\Delta\varepsilon_e}{2(1+\nu)\sigma_e^*}\right)S_{ij}^*} + \left(\sigma_{kk}^n + \frac{E\Delta\varepsilon_{kk}}{(1-2\nu)}\right)\frac{\delta_{ij}}{3}$$

Computing "consistent tangent" DDSDDE (I, J)

$$\frac{\partial \sigma_{ij}^{n+1}}{\partial \Delta \varepsilon_{kl}}$$

Focus on $S_{ij}^{n+1} = \left(1 - \dfrac{3E\Delta\varepsilon_e}{2(1+\nu)\sigma_e^*}\right) S_{ij}^*$

$$\frac{\partial S_{ij}^{n+1}}{\partial \Delta \varepsilon_{kl}} = \left(1 - \frac{3E\Delta\varepsilon_e}{2(1+\nu)\sigma_e^*}\right)\frac{\partial S_{ij}^*}{\partial \Delta \varepsilon_{kl}}$$

$$- \frac{3E}{2(1+\nu)}\left\{\frac{\partial \Delta\varepsilon_e}{\partial \sigma_e^*} - \frac{\Delta\varepsilon_e}{\sigma_e^*}\right\}\frac{1}{\sigma_e^*}\frac{\partial \sigma_e^*}{\partial S_{pq}^*}\frac{\partial S_{pq}^*}{\partial \Delta \varepsilon_{kl}} S_{ij}^*$$

To find $\dfrac{\partial \Delta\varepsilon_e}{\partial \sigma_e^*}$ recall

3. Using Newton-Raphson iteration, solve for $\Delta\varepsilon_e$

$$\sigma_e^* - Y\left(1+\frac{\varepsilon_e^n+\Delta\varepsilon_e}{\varepsilon_0}\right)^{1/n}\left(\frac{\Delta\varepsilon_e}{\Delta t\dot{\varepsilon}_0}\right)^{1/m} - \frac{3E}{2(1+\nu)}\Delta\varepsilon_e = 0$$

$$\Rightarrow \frac{d\sigma_e^*}{d\Delta\varepsilon_e} = \frac{3E}{2(1+\nu)} + Y\left(1+\frac{\varepsilon_e+\Delta\varepsilon_e}{\varepsilon_0}\right)^{1/n}\left(\frac{\Delta\varepsilon_e}{\Delta t\dot{\varepsilon}_0}\right)^{1/m}\left\{\frac{1}{m\Delta\varepsilon_e}+\frac{1}{n(\varepsilon_0+\varepsilon_e+\Delta\varepsilon_e)}\right\}$$

Finally $\sigma_e^* = \sqrt{\frac{3}{2}S_{ij}^*S_{ij}^*}$

$$\Rightarrow \frac{\partial\sigma_e^*}{\partial S_{pq}^*} = \frac{3}{2}\frac{S_{pq}^*}{\sigma_e^*}$$

$$\frac{\partial S_{ij}^*}{\partial\Delta\varepsilon_{kl}} = \frac{E}{1+\nu}\left(\frac{\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}}{2} - \frac{1}{3}\delta_{ij}\delta_{kl}\right)$$

Combine (and use $S^*_{kk} = 0$)

$$\gamma = \frac{3E}{2(1+\nu)\sigma_e^*} + \left(1 - \frac{3E\Delta\varepsilon_e}{2(1+\nu)\sigma_e^*}\right)\left(\frac{1}{n(\varepsilon_0 + \varepsilon_e + \Delta\varepsilon_e)} + \frac{1}{m\Delta\varepsilon_e}\right)$$

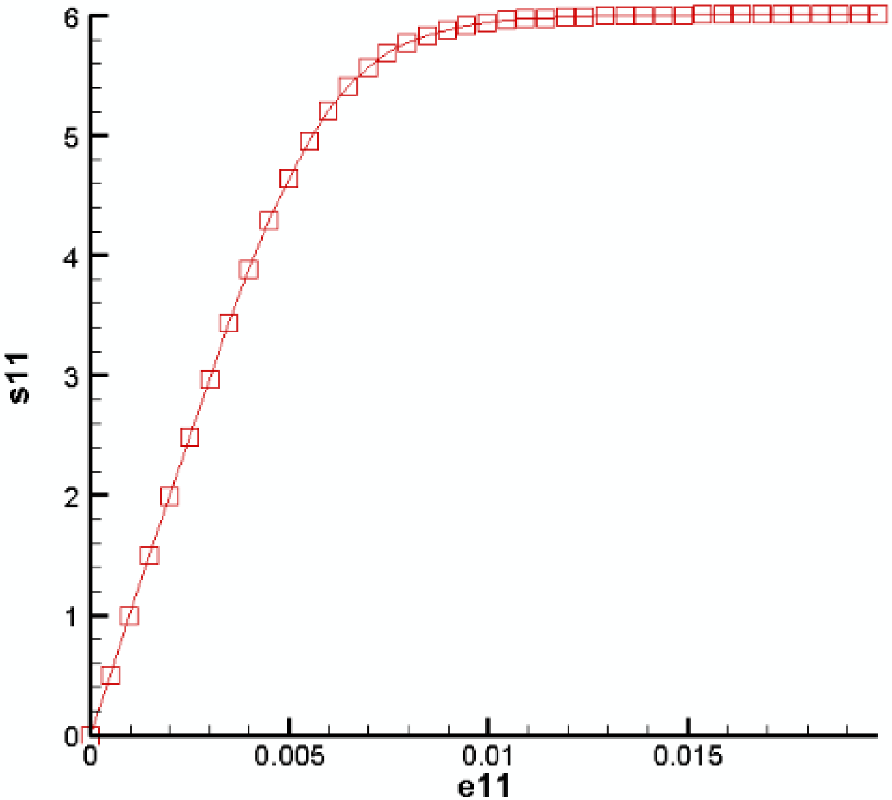
$$\frac{\partial\sigma_{ij}}{\partial\Delta\varepsilon_{kl}} = C^{ep}_{ijkl} = \frac{E}{1+\nu}\left(1 - \frac{3E\Delta\varepsilon_e}{2(1+\nu)\sigma_e^*}\right)\left(\frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}\right) - \frac{1}{3}\delta_{ij}\delta_{kl}\right) + \frac{E}{(1+\nu)}\frac{9E\left(\Delta\varepsilon_e - 1/\gamma\right)}{4(1+\nu)\sigma_e^*}\frac{S^*_{ij}}{\sigma_e^*}\frac{S^*_{kl}}{\sigma_e^*} + \frac{E}{3(1-2\nu)}\delta_{ij}\delta_{kl}$$

Matrix form (for UMAT)

$$\mathbf{D} = \frac{E}{1+\nu}\left(1 - \frac{3E\Delta\varepsilon_e}{2(1+\nu)\sigma_e^*}\right)\begin{bmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1/2 & & \\ & & & & 1/2 & \\ 0 & & & & & 1/2 \end{bmatrix} + \frac{E}{(1+\nu)}\frac{9E\left(\Delta\varepsilon_e - 1/\gamma\right)}{4(1+\nu)\sigma_e^*}\frac{1}{\sigma_e^{*2}}\underline{S}^* \otimes \underline{S}^* + \frac{1}{3}\left[\frac{E}{(1-2\nu)} - \frac{E}{1+\nu}\left(1 - \frac{3E\Delta\varepsilon_e}{2(1+\nu)\sigma_e^*}\right)\right]\underline{I} \otimes \underline{I}$$

$$\underline{S} = [S^*_{11}, S^*_{22}, S^*_{33}, S^*_{12}, S^*_{13}, S^*_{23}] \qquad \underline{I} = [1,1,1,0,0,0]$$

Representative result (see UMAT code online)



Test with 1 element

Use the "user print" file from L12 to produce the x-y data