Review – Modal Dynamics

Equation of motion $\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F}(t)$

1. Find the eigenvectors/values $\mathbf{q}_i, \omega_i^2$ of dynamical matrix $\mathbf{H} = \mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}$

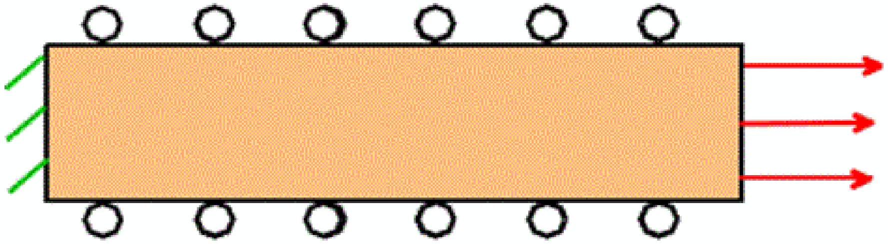
2. Calculate mode shapes $\mathbf{u}_i^* = \mathbf{M}^{-1/2}\mathbf{q}_i$

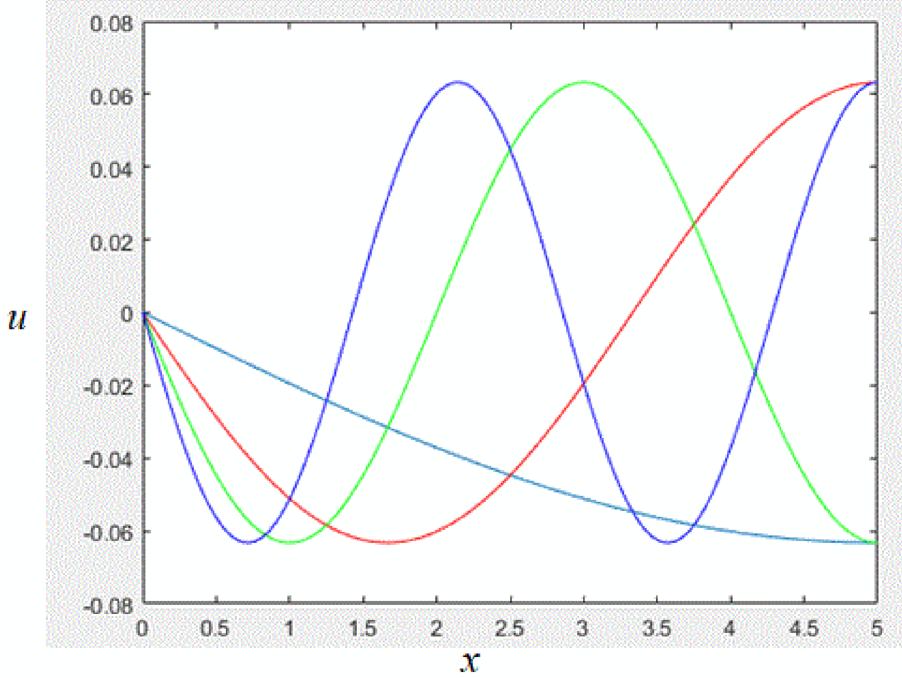
3. Solve decoupled ODEs for the motion of each individual mode

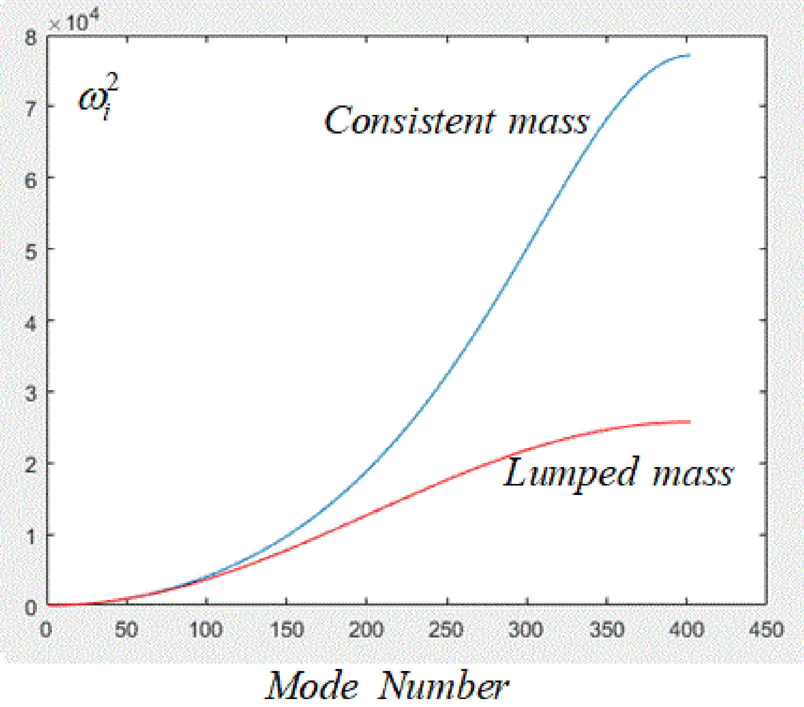
$$\ddot{r}_i(t) + \omega_i^2 r_i(t) = \mathbf{q}_i \cdot \mathbf{M}^{-1/2}\mathbf{F}(t) \qquad r_i(0) = \mathbf{q}_i \cdot \mathbf{M}^{1/2}\mathbf{u}(0) \qquad \dot{r}_i(0) = \mathbf{q}_i \cdot \mathbf{M}^{1/2}\mathbf{v}(0)$$

4. Add the motion of all the modes $\mathbf{u}(t) = \sum_i r_i(t)\mathbf{u}_i^*$

Example : 1-D Bar problem

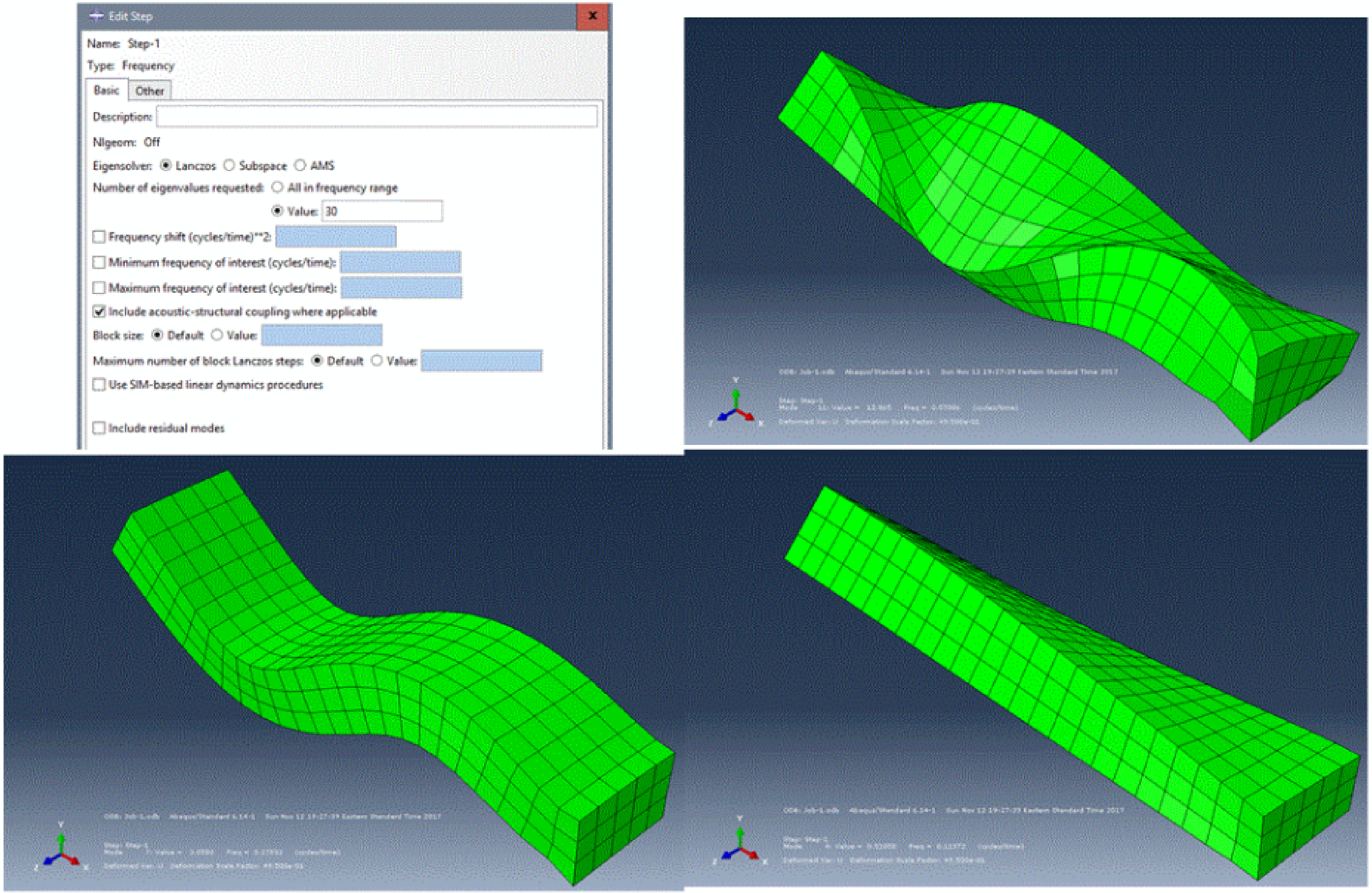






Mode shapes *Frequency spectrum*

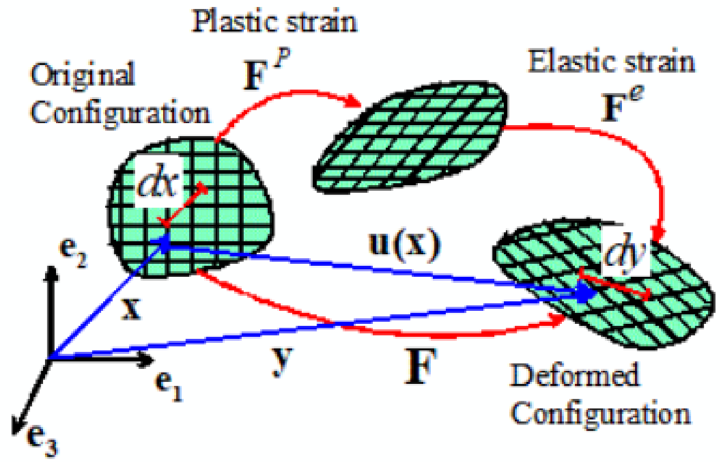
Mode shapes for a cantilever – ABAQUS predictions



9.0 Coding ABAQUS UMAT / VUMAT for finite strain problems

Both subroutines are designed for finite strain plasticity

Background : Finite Strain Plasticity



Deformation $F = I + \nabla \underline{u}$ $\quad J = \det(F)$

Assume $F = F^e F^p$

Elastic + large rotation

Plastic (might include rotation)

- Velocity Gradient $L_{ij} = \dfrac{\partial \dot{u}_i}{\partial y_j} = \dot{F}_{ik} F^{-1}_{kj}$

- Decompose L into symmetric / skew parts

$$L = \dot{\varepsilon} + W \qquad \dot{\varepsilon} = sym(L) \qquad W = skew(L)$$

We can split L, $\dot{\varepsilon}$, W into elastic & plastic parts

$$L = \dot{F}F^{-1} = \frac{d}{dt}(F^eF^p)\,F^{p-1}F^{e-1}$$

$$= \underbrace{\dot{F}^e F^{e-1}}_{L^e} + \underbrace{F^e \dot{F}^p F^{p-1} F^{e-1}}_{L^p}$$

Then let $\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p \qquad \dot{\varepsilon}^e = sym(L^e) \qquad \dot{\varepsilon}^p = sym(L^p)$

$$W = W^e + W^p \qquad W^e = skew(L^e) \qquad W^p = skew(L^p)$$

- Finally define elastic Lagrange strain

$$E^e = \frac{1}{2}\left(F^{eT} F^e - I\right)$$

- Stress measures :

Cauchy Stress σ_{ij} - "true" stress

"Mandel" stress : $\Sigma^e = J\, F^{e-1}\, \sigma\, F^{e-T}$

(Work-conjugate to $\frac{dE^e}{dt}$)

- Constitutive Model

Approximations : Assume $E^e_{ij} E^e_{ij} << 1$
(small elastic strain ; large rotation)
Also $J \approx 1$

Then we can assume elastic strain energy density is approximately

$$\phi = \frac{1}{2} C_{ijkl} E^e_{ij} E^e_{kl}$$

Thermodynamics shows $\Sigma^e_{ij} = \dfrac{\partial \phi}{\partial E_{ij}}$

$$\Sigma^e_{ij} = C_{ijkl} E^e_{kl}$$

Plastic constitutive law (same as small strains)

$$\dot{\varepsilon}^p_{ij} = \dot{\varepsilon}_0 \left(\frac{\sigma_e}{\sigma_0}\right)^m \frac{S_{ij}}{\sigma_e} \qquad S_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$$

$$\sigma_e = \sqrt{3 S_{ij} S_{ij}/2}$$

σ_0 – flow strength

We also need a constitutive equation for W^p

Most models take $W^p = 0$ (not the only choice)

Finite strain plasticity in ABAQUS

Recall: In UMAT/VUMAT:

Given σ_{ij}^{n}, state vari Δt, $\Delta \varepsilon_{ij} = \Delta t\, \dot{\varepsilon}_{ij}$

You must compute σ_{ij}^{n+1}

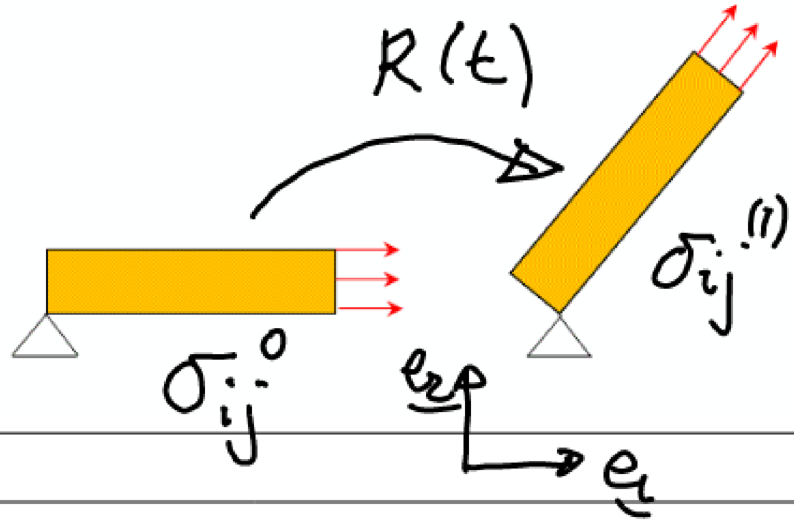
ABAQUS assumes that σ_{ij}^{n+1} can be decomposed as

$$\sigma_{ij}^{n+1} = \sigma_{ij}^{n} + \underbrace{\Delta \sigma_{ij}^{spin}}_{\text{Stress increment from rigid rotation}} + \underbrace{\Delta \sigma_{ij}^{corot}}_{\text{Stress increment from deformation}}$$

ABAQUS computes $\Delta\sigma_{ij}^{spin}$ internally

Understanding stress rates caused by spin

Consider rigid rotation applied to a deformed solid



Let $R(t)$ be rotation tensor

$$W = \dot{R} R^T \qquad W^T = -W$$

$$\sigma^{(1)} = R\,\sigma^{(0)} R^T$$

Now

$$\frac{d\sigma^{(1)}}{dt} = \dot{R}\,\sigma^{(0)} R^T + R\,\sigma^{(0)} \dot{R}^T + R\,\dot{\sigma}^{(0)} R^T$$

$$= \dot{R} R^T R\,\sigma^{(0)} R^T + R\,\sigma^{(0)} R^T R\,\dot{R}^T + R\,\dot{\sigma}^{(0)} R^T$$

$$= W\sigma^{(1)} - \sigma^{(1)} W + R\,\dot{\sigma}^{(0)} R^T$$

$$\frac{d\sigma^{(1)}}{dt} = \underbrace{W\sigma^{(1)} - \sigma^{(1)}W}_{\text{Stress rate caused by spin}} + \underbrace{R\,\mathring{\sigma}^{(0)}R^T}_{\text{Co-rotational stress rate}}$$

ABAQUS stress update :

Given σ_{ij}^n from end of preceding step

(1) Estimate W

(2) Compute $\tilde{\sigma}_{ij} = \sigma_{ij}^n + \Delta t\,(W\sigma_{ij}^n - \sigma_{ij}^n W)$

(3) $\tilde{\sigma}_{ij}$ passed to UMAT or VUMAT.

(4) User computes $\sigma_{ij}^{n+1} = \tilde{\sigma}_{ij} + \Delta \overset{\nabla}{\sigma}_{ij}$ corotational

(5) User computes $\frac{\partial \Delta \overset{\nabla}{\sigma}_{ij}}{\partial \Delta \varepsilon_{kl}}$ (part of tangent)

(6) ABAQUS adds additional terms from spin to tangent

(7) Compute internal force, stiffness etc.

To implement this, we need to define "rigid rotation" part of F so we can compute W

- Depends on choice of material model
- ABAQUS uses a built in default that is consistent with finite strain plasticity