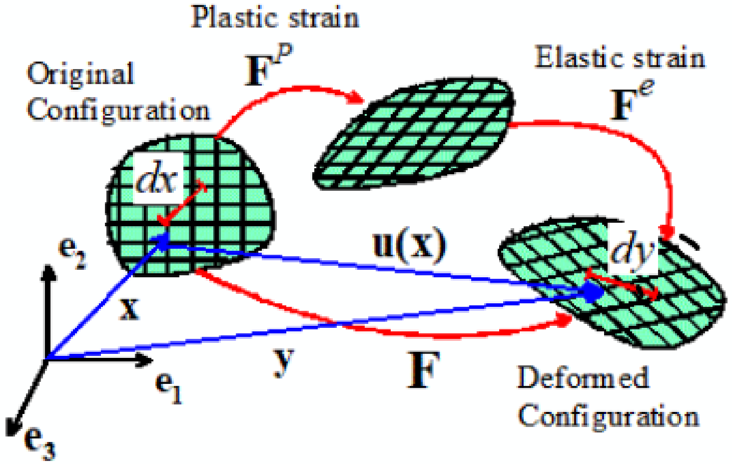
Review – Finite strain plasticity

$$F_{ij} = F^e_{ik} F^p_{kj} \qquad L_{ij} = \dot{F}_{ik} F^{-1}_{kj} = \dot{F}^e_{ik} F^{e-1}_{kj} + F^e_{ik} \dot{F}^p_{kl} F^{p-1}_{lm} F^{e-1}_{mj}$$

$$\dot{\varepsilon}^e_{ij} = sym(\dot{F}^e_{ik} F^{e-1}_{kj}) \qquad W^e_{ij} = skew(\dot{F}^e_{ik} F^{e-1}_{kj})$$

$$\dot{\varepsilon}^p_{ij} = sym(F^e_{ik} \dot{F}^p_{kl} F^{p-1}_{lm} F^{e-1}_{mj}) \quad W^p_{ij} = skew(\dot{F}^p_{ik} F^{p-1}_{kj})$$

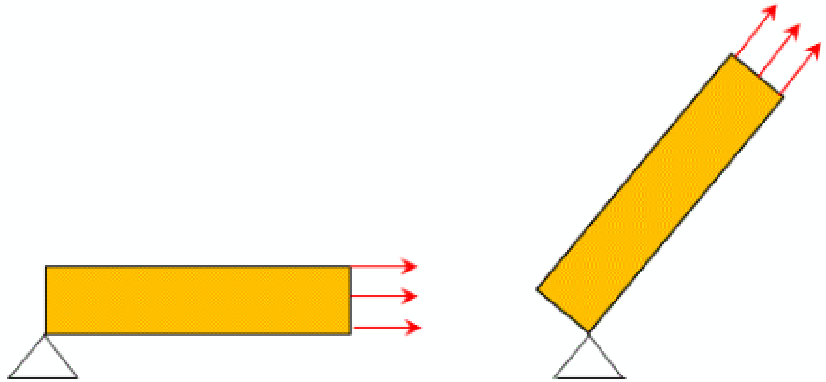


Material model:

Plasticity: $\dot{\varepsilon}^p_{ij} = \dot{\varepsilon}_0 \left(\dfrac{\sigma_e}{\sigma_0} \right)^m \dfrac{3}{2} \dfrac{S_{ij}}{\sigma_e} \qquad S_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3 \qquad \sigma_e = \sqrt{3 S_{ij} S_{ij}/2} \qquad W^p_{ij} = 0$

Elasticity: $\Sigma^e_{ij} = C_{ijkl} E^e_{kl} \qquad E^e_{ij} = (F^e_{ki} F^e_{kj} - \delta_{ij})/2 \qquad \Sigma^e_{ij} = J F^{e-1}_{ik} \sigma_{kl} F^{e-1}_{lj}$

Review – Stress Update in ABAQUS



Stress rate is assumed to have form:

$$\frac{d\sigma_{ij}}{dt} = \overset{\nabla corot}{\sigma_{ij}} - \sigma_{ik}W_{kj} + W_{ik}\sigma_{kj}$$

Stress update:

$$\sigma_{ij}^{n+1} = \sigma_{ij}^{n} + \Delta\left(W_{ik}\sigma_{kj} - \sigma_{ik}W_{kj}\right) + \Delta\overset{\nabla corot}{\sigma_{ij}} = \underbrace{\Delta R_{ik}\sigma_{kl}^{n}\Delta R_{jl}}_{\text{Computed by ABAQUS:}} + \underbrace{\Delta\overset{\nabla corot}{\sigma_{ij}}}_{\text{Updated in UMAT/VUMAT}}$$

Procedure:

1. ABAQUS computes $\tilde{\sigma}_{ij} = \Delta R_{ik}\sigma_{kl}\Delta R_{jl}$
2. Input to UMAT/VUMAT is $\tilde{\sigma}_{ij}$
3. User computes $\sigma_{ij}^{n+1} = \tilde{\sigma}_{ij} + \Delta\overset{\nabla corot}{\sigma_{ij}}$ (and $\dfrac{\partial\Delta\overset{\nabla corot}{\sigma_{ij}}}{\partial\Delta\varepsilon_{kl}}$ if UMAT)
4. ABAQUS/Standard adds additional term to tangent

Choice of spin W_{ij} in ABAQUS

ABAQUS / Standard with UMAT

W_{ij} is the skew part of $L_{ij} = \dot{F}_{ik} F_{kj}^{-1}$

ABAQUS / explicit with VUMAT

W_{ij} is $\dot{R} R^T$ where R is calculated using polar decomposition $F = RU$

Note that choice of W may change if

(1) Local orientations are used

(2) for structural elements rotation is computed from rotation of x-sect

Which choice of W is correct?

- Depends on material model

For finite strain plasticity model

Recall $\quad \varepsilon^e_{ij} = C_{ijkl} E^e_{kl} \qquad J\sigma = F^e \varepsilon^e F^{eT}$

Hence $\quad \dfrac{d(J\sigma)}{dt} = \dot{F}^e \varepsilon F^{eT} + F^e \varepsilon \dot{F}^{eT} + F^e \dot{\varepsilon}^e F^{eT}$

$$= \dot{F}^e F^{e-1}(J\sigma) + (J\sigma) F^{e-T} \dot{F}^{eT} + F^e \dot{\varepsilon}^e F^{eT}$$

Finally $\quad \dot{F}^e F^{e-1} = \dot{\varepsilon}^e + W^e = \dot{\varepsilon}^e + W - W^p$

$$\frac{d(J\sigma)}{dt} = \underbrace{W J\sigma - J\sigma W}_{\overset{\circ}{\sigma}^{spin}} + \overbrace{F^e \dot{\varepsilon}^e F^{eT} + \dot{\varepsilon}^e J\sigma + J\sigma \dot{\varepsilon}^e}^{\overset{\nabla}{\sigma}^{convt}} \underbrace{- W^p (J\sigma) + J\sigma (W^p)}_{\text{-from plastic spin - usually } 0}$$

As long as $W^P = 0$ this has form

$$\frac{d(J\sigma)}{dt} = \dot{\sigma}^{spin} + \overset{\nabla corot}{\sigma}$$

$\dot{\sigma}^{spin}$ is consistent with ABAQUS/standard

Not consistent with explicit & VUMAT

Differences greatest for deformation consisting of simultaneous shear & rotation

Expression for $\overset{\nabla corot}{\sigma}$

Consider $F^e \dot{\Sigma}^e F^{eT}$ $\qquad \Sigma_{ij} = C_{ijkl} E^e_{kl}$

Hence $\dot{\Sigma}^e = C : \dot{E}^e$ $\qquad$ Recall $\dot{E}^e = F^{eT} \dot{\varepsilon}^e F^e$

Combine : $F^e_{ip}\dot{\varepsilon}^e_{pq}F^e_{jq} = F^e_{ip}F^e_{jq}C_{pqmn}F^e_{km}F^e_{ln}\dot{\varepsilon}^e_{kl}$

or $$F^e\dot{\varepsilon}^eF^{eT} = \underbrace{\left[F^eF^eCF^{eT}F^{eT}\right]}_{\substack{\text{Interpret as elastic moduli} \\ \text{rotated to current config } \tilde{C}}} : \dot{\varepsilon}^e$$

For isotropic C we can approximate

$$F^eF^eCF^{eT}F^{eT} \approx C$$

$$\overset{\nabla corot}{(J\sigma)} = \tilde{C} : \dot{\varepsilon}^e + \underbrace{(J\sigma)\dot{\varepsilon}^e + \dot{\varepsilon}^e(J\sigma)}_{\substack{\text{Usually small compared} \\ \text{to first term}}}$$

Hence $\overset{\nabla corot}{(J\sigma)} \approx \tilde{C} : \dot{\varepsilon}^e$ — small-strain form

Final remarks

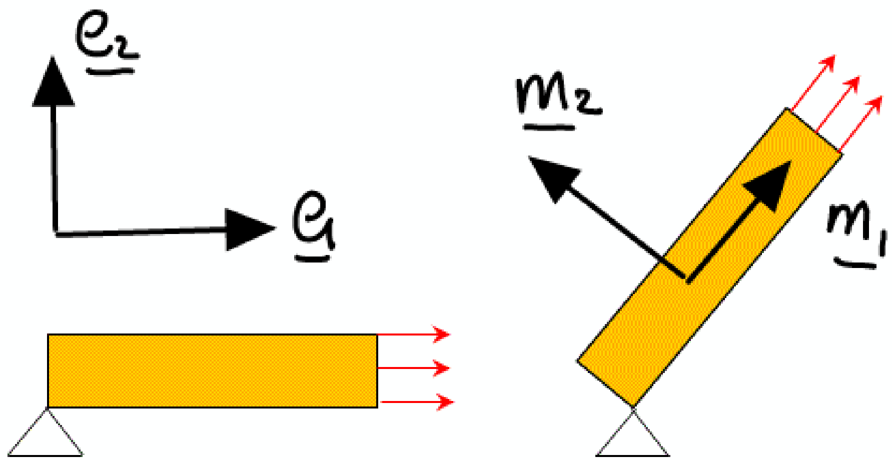
(1) Precise expression for stress rate will depend on choice of material model

- We can derive a rate form for any material model
- Result may not be consistent with ABAQUS assumption
- Can always switch to a different choice of $\dot{\sigma}^{spin}$ in UMAT – just remove ABAQUS correction

(2) Conventions for stress components in VUMAT

In UMAT all tensors are in global basis

In VUMAT several tensor components are expressed in a rotating basis



$$\sigma = \sigma_{ij}\, \underline{e}_i \otimes \underline{e}_j$$
$$= \hat{\sigma}_{ij}\, \underline{m}_i \otimes \underline{m}_j$$

VUMAT provides $\hat{\sigma}_{ij}$ $\Delta \hat{\varepsilon}_{ij}$ $\hat{V}_{ij}$ in rotating basis

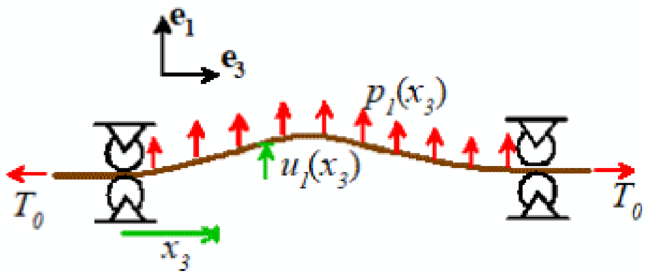
ABAQUS assumes $\underline{m}_i = R\, \underline{e}_i$ where $F = RU = VR$

10) Structural Elements

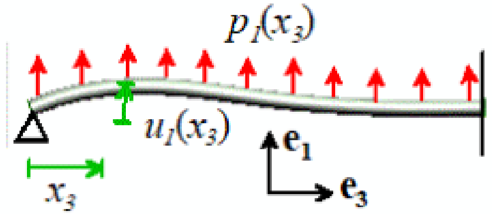
Goal: Develop efficient FEA for solids with special geometries - beams, plates, shells

Two general approaches :

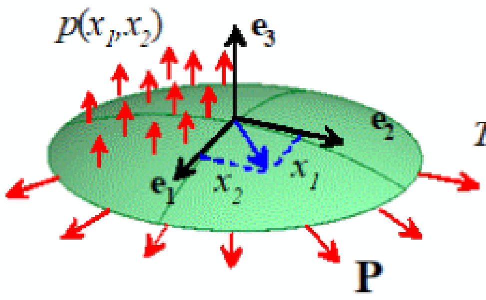
(1) Get weak form and interpolate classical theories



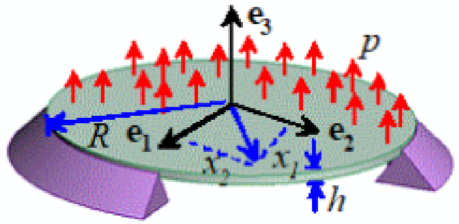
$$T_0 \frac{d^2 u_1}{dx_3^2} + p_1 = \rho A \frac{d^2 u_1}{dt^2}$$



$$EI_2 \frac{d^4 u_1}{dx_3^4} + \rho A \frac{d^2 u_1}{dt^2} = p_1$$



$$T_{\alpha\beta} \frac{\partial^2 u_3}{\partial x_\alpha \partial x_\beta} + p_3 = \rho h \frac{\partial^2 u_3}{dt^2}$$



$$\frac{Eh^3}{12(1-\nu^2)} \frac{\partial^4 u_3}{\partial x_\alpha \partial x_\alpha \partial x_\beta \partial x_\beta} + \rho h \frac{\partial^2 u_3}{dt^2} = p$$

(2) Adapt standard interpolation schemes to be consistent with the assumptions made to derive classical equations

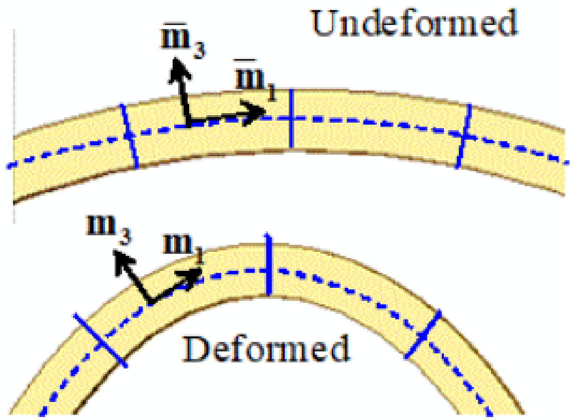
"Continuum" based beams, plates & shells

- Focus on (2) here

Background : kinematic assumptions in structural theories

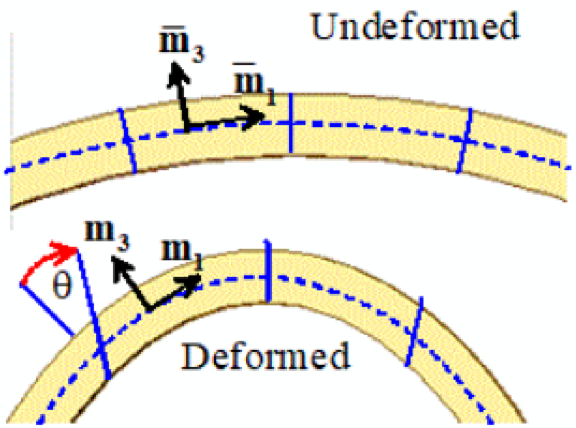
1 Euler-Bernoulli beam (Kirchhoff plate / shell) :

Plane sections transverse to neutral axis remain plane and perpendicular to neutral sect after deformation



(2) Timoshenko beam
(Reissner - Mindlin shell)

Sections normal to neutral section remain flat but can rotate



(2) is better for thicker cross-sections

Continuum elements are similar to Timoshenko / Reissner / Mindlin structures

- will solve for internal shear deformation along with displacements