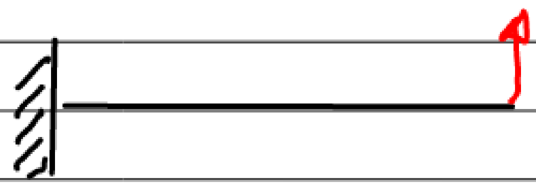
Continuum Beam Elements



Assumptions:

(1) 2D beams
(2) Small strains
(3) Linear elasticity

Approach: Start with standard linear elasticity

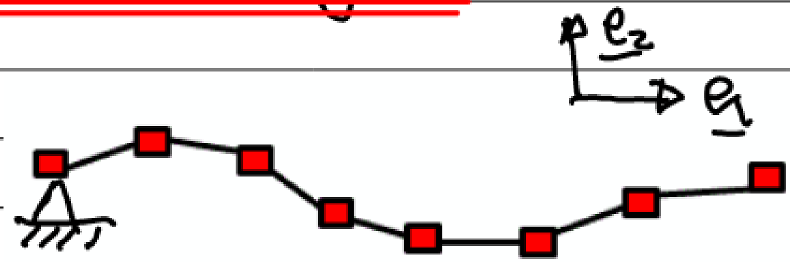
$$[K]\underline{u} = -\underline{R} + \underline{f}$$

$$[K] = \sum_{el} \int_{\Omega_e} [B]^T [D] [B]\, dV \qquad \underline{R} = \sum_{el} \int_{\Omega_e} [B]^T \underline{\sigma}\, dV$$

$$\underline{f} = \int_{S_2} t_i^* \delta u_i \, dA$$

But: (1) Describe defomation using set of reduced DOF
(2) Modify $\underline{\sigma}$ to be consistent with beam theory

Given information in defining 2D beam analysis

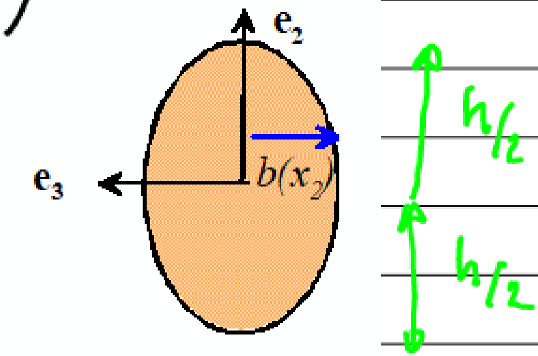


Geometry:

(1) Coords of points on neutral of beam "master nodes"

Props E, μ

(2) Geometry of x-sect $b(x_2)$



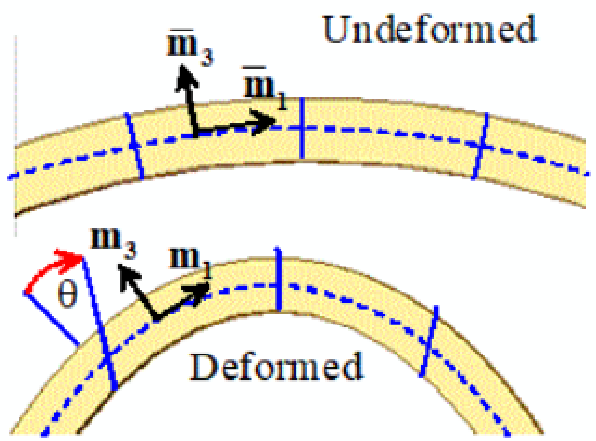
Assume symmetric x-section

Unknowns will be $[u_1, u_2, \delta\theta]$ at each master node

Boundary Conditions: Can prescribe u_i, $\delta\theta$ at any node or $[F_1, F_2, M_3]$ at any node

or distributed pressure on surface

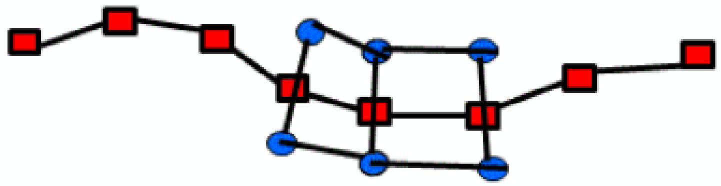
Describing deformations in x-section



Adopt "Timoshenko" type approximation
- cross sections can rotate wrt centerline
- Shear strains exist inside section

$\delta\theta$ represents rotation of x-sect

To calculate strains in the beam



(1) Create two "slave" nodes for each "master" - lie on top & bottom of section

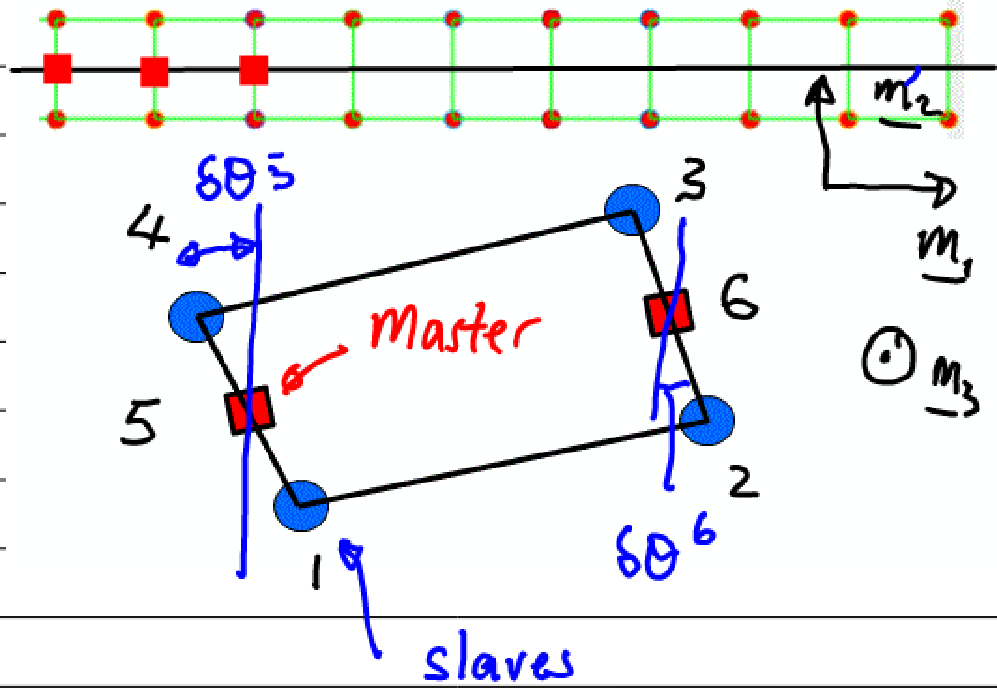
(2) Create a mesh of quadrilateral elements using slaves
- use usual interpolations to find strains

Coordinates of slave nodes:

(1) $\underline{m}_1 = \dfrac{(\underline{x}^6 - \underline{x}^5)}{|\underline{x}^6 - \underline{x}^5|}$

$$\underline{m}_2 = \underline{m}_3 \times \underline{m}_1$$

$$\left.\begin{aligned} \underline{x}^1 &= \underline{x}^5 - \frac{h}{2}\,\underline{m}_2 \\ \underline{x}^4 &= \underline{x}^5 + \frac{h}{2}\,\underline{m}_2 \end{aligned}\right\} \text{etc}$$



Displacements:

Cross product

$$\underline{u}^1 = \underline{u}^5 + \delta\theta^5\,\underline{m}_3 \wedge (\underline{x}^1 - \underline{x}^5)$$

$$\underline{u}^4 = \underline{u}^5 + \delta\theta^5\,\underline{m}_3 \wedge (\underline{x}^4 - \underline{x}^5)$$

This is a linear transformation of form

$$\underline{u}^{slave} = [T]\,\underline{u}^{master}$$

$$\underline{u}^{slave} = [u_1^1 \; u_2^1 \; u_1^2 \; u_2^2 \cdots u_1^4 \; u_2^4]$$

$$\underline{u}^{master} = [u_1^5 \; u_2^5 \; \delta\theta^5 \quad u_1^6 \; u_2^6 \; \delta\theta^6]$$

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & x_2^5 - x_2^1 & 0 & 0 & 0 \\ 0 & 1 & -(x_1^5 - x_1^1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & x_2^6 - x_2^2 \\ 0 & 0 & 0 & 0 & 1 & x_1^6 - x_1^2 \\ 0 & 0 & 0 & 1 & 0 & x_2^6 - x_2^3 \\ 0 & 0 & 0 & 0 & 1 & x_1^6 - x_1^3 \\ 1 & 0 & x_2^5 - x_2^4 & 0 & 0 & 0 \\ 0 & 1 & -(x_1^5 - x_1^4) & 0 & 0 & 0 \end{bmatrix}$$

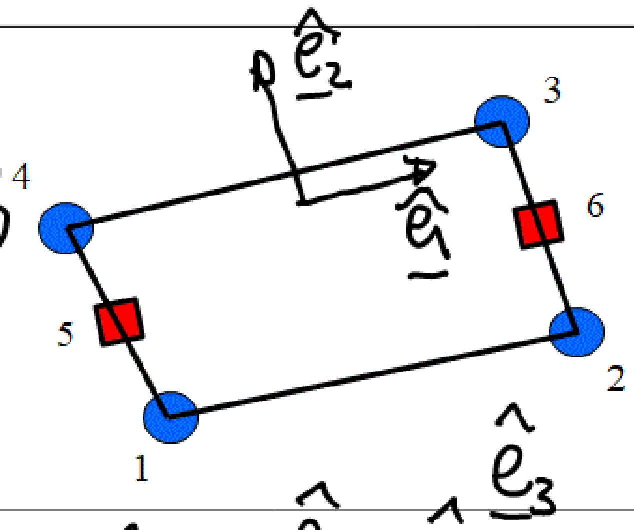
Hence we can calculate strains using usual 4 noded interpolation between slave nodes

$$\underline{\varepsilon} = [B]\,\underline{u}^{slave} = [B][T]\,\underline{u}^{master}$$

[B] for 2D quadrilateral

$$\underline{\varepsilon} = [\varepsilon_{11} \quad \varepsilon_{22} \,, 2\varepsilon_{12}]$$

Calculating stresses



Introduce "Laminar coordinates" $\{\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3\}$

Let $-1 < \xi_i < +1$ be the normalized coords for element

$$\underline{e}_1 = \frac{1}{\left|\frac{\partial \underline{x}}{\partial \xi_1}\right|} \frac{\partial \underline{x}}{\partial \xi_1} \qquad \underline{e}_2 = \frac{1}{\left|\frac{\partial \underline{x}}{\partial \xi_2}\right|} \frac{\partial \underline{x}}{\partial \xi_2} \qquad \underline{e}_3 = \hat{\underline{e}}_1 \times \hat{\underline{e}}_2$$

Beam theory assumes that $\hat{\sigma}_{11}$ $\hat{\sigma}_{12}$ are the only nonzero stresses and are related to strains by

$$\hat{\underline{\sigma}} = [D]\, \hat{\underline{\varepsilon}} \qquad \hat{\underline{\sigma}} = [\hat{\sigma}_{11} \;\; \hat{\sigma}_{12}]$$

$$\hat{\underline{\varepsilon}} = [\hat{\varepsilon}_{11}, 2\hat{\varepsilon}_{12}]^T$$

$$[D] = \begin{bmatrix} E & 0 \\ 0 & \mu \end{bmatrix}$$

We need to transform $\underline{\varepsilon}$ (in global basis) to $\hat{\varepsilon}$ (laminar basis)

$$\varepsilon = \varepsilon_{ij}\, \underline{e}_i \otimes \underline{e}_j \qquad \hat{\varepsilon} = \hat{\varepsilon}_{ij}\, \hat{e}_i \otimes \hat{e}_j$$

$$\varepsilon = \hat{\varepsilon} \Rightarrow \varepsilon_{ij}\, \underline{e}_i \otimes \underline{e}_j = \hat{\varepsilon}_{ij}\, \hat{e}_i \otimes \hat{e}_j$$

$$\Rightarrow \hat{\varepsilon}_{kl} = \varepsilon_{ij} (\hat{\underline{e}}_k \cdot \underline{e}_i)(\underline{e}_j \cdot \hat{e}_l)$$

Linear transformation $\quad \underline{\hat{\varepsilon}} = [R]\, \underline{\varepsilon}$

$$\mathbf{R} = \begin{bmatrix} (\hat{\mathbf{e}}_1 \cdot \mathbf{e}_1)^2 & (\hat{\mathbf{e}}_2 \cdot \mathbf{e}_2)^2 & (\hat{\mathbf{e}}_1 \cdot \mathbf{e}_1)(\hat{\mathbf{e}}_1 \cdot \mathbf{e}_2) \\ 2(\hat{\mathbf{e}}_1 \cdot \mathbf{e}_1)(\hat{\mathbf{e}}_2 \cdot \mathbf{e}_1) & 2(\hat{\mathbf{e}}_1 \cdot \mathbf{e}_2)(\hat{\mathbf{e}}_2 \cdot \mathbf{e}_2) & (\hat{\mathbf{e}}_1 \cdot \mathbf{e}_1)(\hat{\mathbf{e}}_2 \cdot \mathbf{e}_2) + (\hat{\mathbf{e}}_1 \cdot \mathbf{e}_2)(\hat{\mathbf{e}}_2 \cdot \mathbf{e}_1) \end{bmatrix}$$

Hence $\hat{\sigma} = [D]\,[R]\,[B]\,[T]\,\underline{u}^{master}$

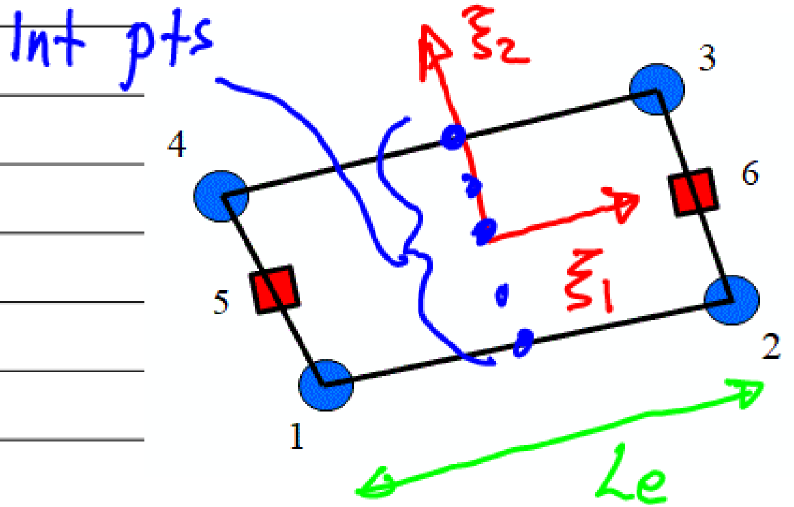
Finally we can substitute these back into standard formula for internal force vec & stiffness

$$[K]\,\underline{u}^{master} = -\underline{R} + \underline{f}$$

$$\underline{R} = \sum_{elements} \int_{\Omega_{el}} \left[[R][B][T]\right]^T \underline{\hat{\sigma}}\, dV$$

$$[K] = \sum_{elem} \int_{\Omega_{el}} \left[[R][B][T]\right]^T [D]\,[R]\,[B]\,[T]\, dV$$

Evaluating volume integrals



Two issues: (1) need to integrate cross-sectional geometry
(2) Standard quadrature causes serious locking

Assume x-sect is constant along element

$$\int_{\Omega_{el}} dV = \int_0^L ds \int_A dA = L_e \int_A dA$$

Evaluate $\int_A dA$ by integrating along line along $\xi_1 = 0$

$$\int_{\Omega_{el}} (\,)\,dV = L_e \frac{h}{2} \int_{-1}^{+1} (\,)\, 2b\left(\frac{h}{2}\xi_2\right) d\xi_2$$

Finally the integral $\int_{-1}^{+1} d\xi_2$ is integrated by quadrature - could use Gauss quadrature

Usually we use simple 5 point trapezoidal integration

$$\int_{-1}^{+1} f(\xi_2)\, d\xi_2 = \sum_{\text{int pts}} f(\xi_i)\, W_i$$

Summary:

Operations to set up element stiffness & force.

(1) Use coords of master nodes to find $\underline{m}_1, \underline{m}_2$

(2) Set up slave node coords

(3) Find $[T]$ $[D]$

(4) Loop over 1D trapezoidal int pts

(a) Get $\hat{\underline{e}}_i$ (laminar basis vecs)

(b) $[R]$, $[B]$ $\hat{\sigma} = [D][R][B][T]\underline{u}^{master}$

(c) Add contribution $\underline{r}^{el} = \underline{r}^{el} - [R\,B\,T]^T \hat{\sigma}\, w_i$

$$[K^{el}] = [K^{el}] + [RBT]^T [D] RBT W_i$$

end loop