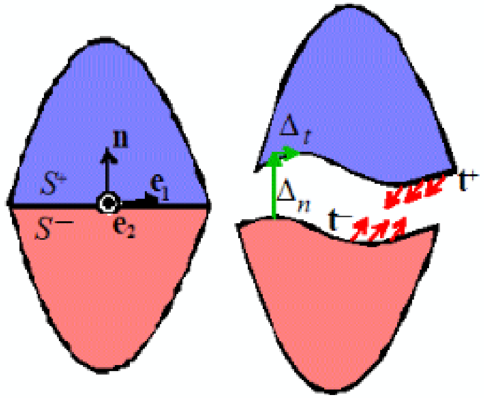
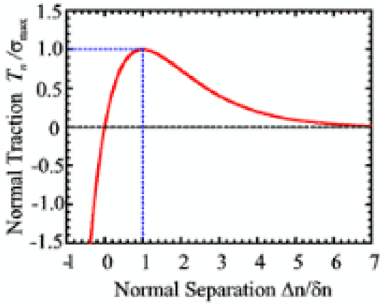
Review – FEA with cohesive zones

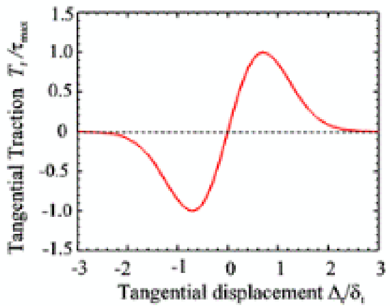
Cohesive zones $T_n = \dfrac{\partial \Phi}{\partial \Delta_n}$ $T_1 = \dfrac{\partial \Phi}{\partial \Delta_1}$ $T_2 = \dfrac{\partial \Phi}{\partial \Delta_2}$



1. Reversible:

$$\Phi(\Delta_n, \Delta_t) = \phi_n - \phi_n \left(1 + \frac{\Delta_n}{\delta_n}\right) \exp\left(-\frac{\Delta_n}{\delta_n}\right) \exp\left(-\frac{\beta^2 \Delta_t^2}{\delta_n^2}\right)$$

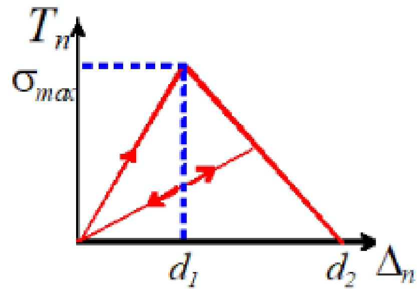




2. Irreversible: $\Phi = k_0 (1 - D) \lambda^2 / 2$

$$\lambda = \sqrt{\Delta_n^2 + \beta^2 (\Delta_1^2 + \Delta_2^2)}$$

$$\frac{dD}{dt} = \begin{cases} 0 & \lambda < \dfrac{d_1 d_2}{(1-D) d_2 + D d_1} \quad \text{or} \quad d\lambda / dt < 0 \quad \text{or} \quad D = 1 \\ \left((1-D) + \dfrac{d_1}{d_2 - d_1}\right) \dfrac{1}{\lambda} \dfrac{d\lambda}{dt} & \text{Otherwise} \end{cases}$$



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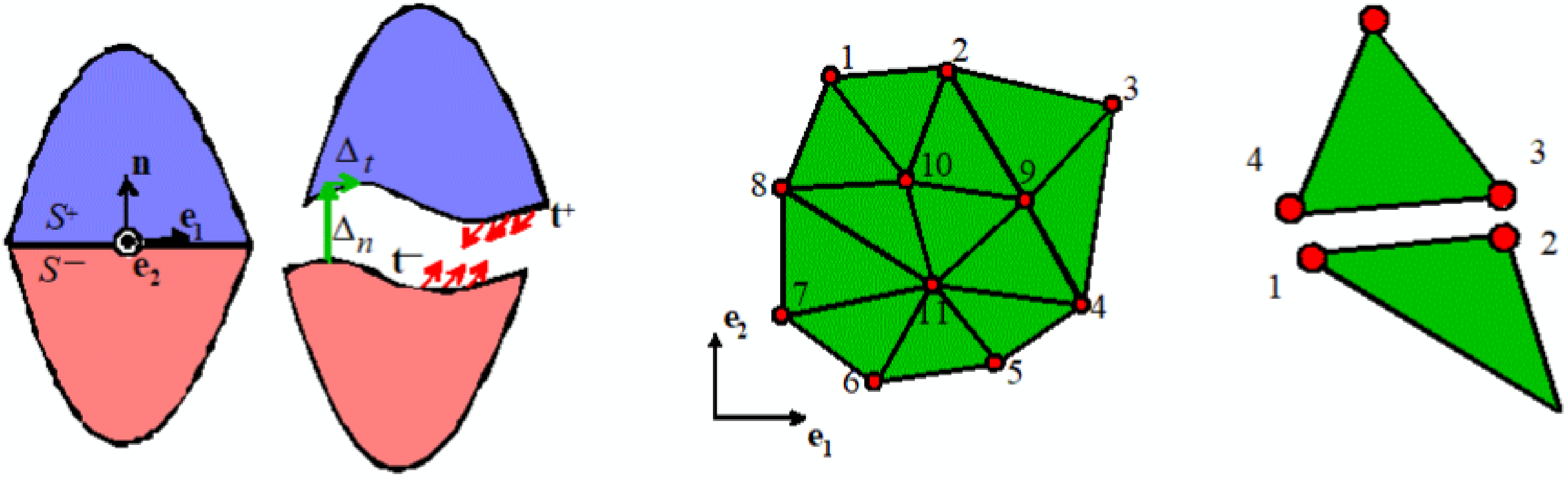


Modified VWE:

$$\int_{V_0} \sigma_{ij}\frac{\partial \eta_i}{\partial x_j}dV + \int_{\Gamma} T_n n_i[\eta_i^+ - \eta_i^-] + \int_{\Gamma} T_\alpha e_i^\alpha[\eta_i^+ - \eta_i^-] = \int_{S_2} t_i^* \eta_i dA$$

Interpolation: $u_i(\mathbf{x}) = \sum_{a=1}^{n} N^a(\mathbf{x})u_i^a \qquad \eta_i(\mathbf{x}) = \sum_{a=1}^{n} N^a(\mathbf{x})\eta_i^a$

Implementing CZ

Discrete equations $R_i^a [u_k^b] = f_i^a$

$$R_i^a = \sum_{el} \int_{\Omega_{el}} \sigma_{ij} [u_k^b] \frac{\partial N^a}{\partial x_j}$$

$$\sum_{el} \int_{\Gamma_{el}} T_n \, n_i \, [N^{a+} - N^{a-}] \, dA$$

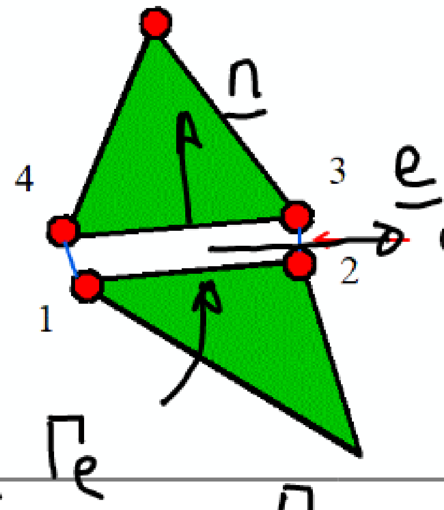
$$\sum_{el} \int_{\Gamma_{el}} T_\alpha \, e_i^\alpha \, [N^{a+} - N^{a-}] \, dA$$

Implement with CZ elements

Solve with Newton - Raphson : Correction dw_k^b

$$K_{aibk} \, dw_k^b = -R_i^a + f_i^a$$

Cohesive Zone Elements



Δ_n, Δ_t } Displacement jumps

Consider 2D element

$$\underset{\sim}{e} = \frac{\underset{\sim}{x}^2 - \underset{\sim}{x}^1}{|\underset{\sim}{x}^2 - \underset{\sim}{x}_i^1|} \qquad \underset{\sim}{n} = \underset{\sim}{k} \times \underset{\sim}{e}$$

Let $-1 < \xi < +1$ parameterize position on Γ_e

Let $N^1 = (\xi - 1)/2 \qquad N^2 = -(\xi+1)/2$

$N^3 = (\xi+1)/2 \qquad N^4 = -(\xi-1)/2$

be interpolation functions.

Define $[B]$ such that $\begin{bmatrix} \Delta_n \\ \Delta_t \end{bmatrix} = [B]\, \underset{\sim}{u}^{el}$

$$[\tilde{B}] = \begin{bmatrix} N^1 n_1 & N^1 n_2 & N^2 n_1 & N^2 n_2 & \cdots & N^4 n_1 & N^4 n_2 \\ N^1 e_1 & N^1 e_2 & N^2 e_1 & N^2 \underline{e_2} & & N^4 \underline{e_1} & N^4 \underline{e_2} \end{bmatrix}$$

Define $\underline{T} = (T_n, T_t)$

Element length

Element residual

$$\underline{r}^{el} = -\int_{\Gamma_{el}} [B]^T \underline{T}\, dA = -\frac{L}{2}\int_{-1}^{+1} [B]^T \underline{T}\, d\xi$$

$$\approx -\underbrace{\sum_{\substack{int \\ pts}} [B]^T \underline{T}\, \frac{L}{2} W_i}_{\text{Use 2 point Gauss}}$$

Element stiffnes

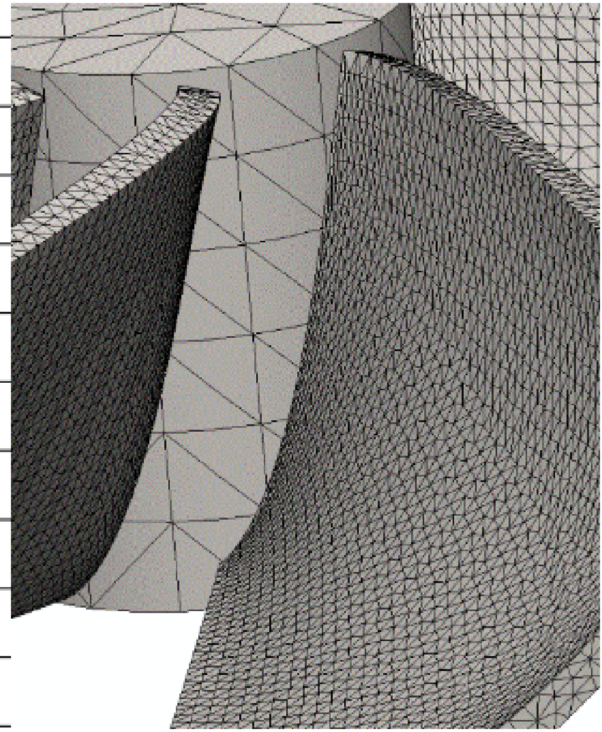
$$[k^{el}] = \frac{h}{2}\int_{-1}^{+1} [B]^T [D] [B] \, d\xi$$

$$[D] = \begin{bmatrix} \frac{\partial T_n}{\partial \Delta_n} & \frac{\partial T_n}{\Delta_t} \\ \frac{\partial T_t}{\partial \Delta_n} & \frac{\partial T_t}{\partial \Delta_t} \end{bmatrix}$$

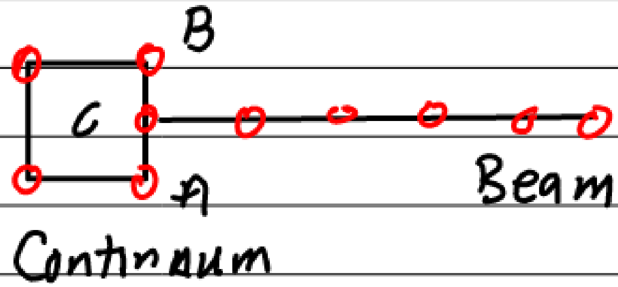
12) Constraints in FEA

Examples of constraints

① Connect incompatible meshes



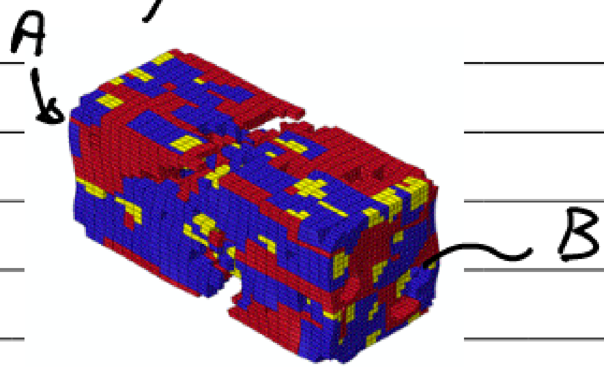
② Connecting different element types



Displacements of A, B related to displacement & rotation of C

③ Periodic boundary conditions

$$u_i^B - u_i^A = E_{ij}\,(x_j^B - x_j^A)$$



Incorporating constraints in FEA analysis

Consider simple constraint on 1 DOF $\quad u_m^A - u_m^B = 0$

Method ① : Penalty method connect DOF with stiff spring

Modified PVW

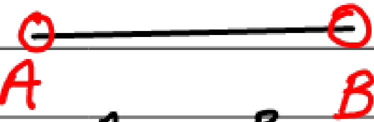
Big spring stiffness

$$\int_R \sigma_{ij} \frac{\partial \eta_i}{\partial y_j} dV + \underbrace{\xi \left(u_m^A - u_m^B \right)}_{\text{Tension}} \underbrace{\left(\eta_m^A - \eta_m^B \right)}_{\text{Extension rate}} = \int_{S_2} t_i^* \eta_i \, dA$$

$$\forall \eta_i$$

We can add the additional term to PVW with a "spring" element

Element force & stiffness



$$\underline{r}^{el} = -\begin{bmatrix} \xi(u_1^A - u_1^B) \\ 0 \\ -\xi(u_1^A - u_1^B) \\ 0 \end{bmatrix} \qquad \underline{u}^{el} = [u_1^A \; u_2^A \; u_1^B \; u_2^B]$$

To constrain DOF #1

$$[k^{el}] = \begin{bmatrix} \xi & 0 & -\xi & 0 \\ 0 & 0 & 0 & 0 \\ -\xi & 0 & \xi & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Reasonably successful ; only option in explicit dynamics

Choice of ξ : Too big $\Rightarrow$ rounding error
Too small $\Rightarrow$ constraint violated

Method ② : Lagrange multipliers

Review: To minimize $\phi(x,y)$ subject to $f(x,y)=0$

To enforce constraint we make $\phi(x,y) + \lambda f(x,y) = 0$ stationary with respect to x, y, λ

$$\Rightarrow \left.\begin{aligned} \frac{\partial \phi}{\partial x} + \lambda \frac{\partial f}{\partial x} &= 0 \\ \frac{\partial \phi}{\partial y} + \lambda \frac{\partial f}{\partial y} &= 0 \\ f(x,y) &= 0 \end{aligned}\right\} \text{3 eqs for } x, y, \lambda$$

Apply to FEA

Consider constraint $f(\underline{u}^A, \underline{u}^B) = 0$

Augment PVW:

$$\int_R \sigma_{ij} \frac{\partial \eta_i}{\partial x_j}\, dV + \underbrace{\delta[\lambda f(\underline{u}^A - \underline{u}^B]}_{\text{Variation wrt } \lambda,\ \underline{u}^A,\ \underline{u}^B} = \int_{S_2} t_i^* \eta_i\, dA$$

$$\delta[\lambda f(\ )] = f(\underline{u}^A\ \underline{u}^B)\, \delta\lambda + \lambda\left(\frac{\partial f}{\partial u_i^A} \eta_i^A - \frac{\partial f}{\partial u_i^B} \eta_i^B\right)$$

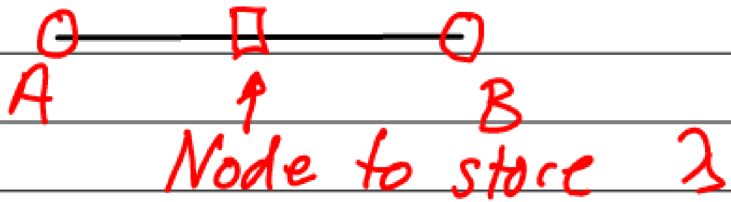
Satisfy augmented PVW $\forall\ \eta_i$ and $\delta\lambda$

Implementation

- We can interpret $\lambda \frac{\partial f}{\partial u_i^A}$ as "force" on node A to satisfy constraint

$$-\lambda \frac{\partial f}{\partial u_i^B} \text{ is force on } B$$

Satisfy with 3 noded Lagrange multiplier element



DOF are: $[u_1^A \; u_2^A \; u_1^B \; u_2^B \; \lambda]$ (2D)

$$\underline{r}^{el} = -\begin{bmatrix} \lambda \dfrac{\partial f}{\partial u_1^A} \\ \lambda \dfrac{\partial f}{\partial u_2^A} \\ -\lambda \dfrac{\partial f}{\partial u_1^B} \\ -\lambda \dfrac{\partial f}{\partial u_2^B} \\ f(\underline{u}^A, \underline{u}^B) \end{bmatrix}$$

$$[K^{el}] = \begin{bmatrix} \lambda \partial^2 f/\partial {u_1^A}^2 & \lambda \dfrac{\partial^2 f}{\partial u_1^A \partial u_2^A} & \cdots & \lambda \dfrac{\partial^2 f}{\partial u_1^A \partial u_1^B} & \cdots & \dfrac{\partial f}{\partial u_1^A} \\ \lambda \dfrac{\partial^2 f}{\partial u_2^A \partial u_1^A} & \cdots & & & & \dfrac{\partial f}{\partial u_2^A} \\ \vdots & & & & & \\ \partial f/\partial u_1^A & \partial f/\partial u_2^A & \cdots & & & 0 \end{bmatrix}$$

For linear constraints 2nd derivatives all vanish

This enforces constraint exactly

Note zero on diagonal : causes problems for some solvers

For explicit dynamics, computing λ always needs some equation solving

- other methods can be preferable