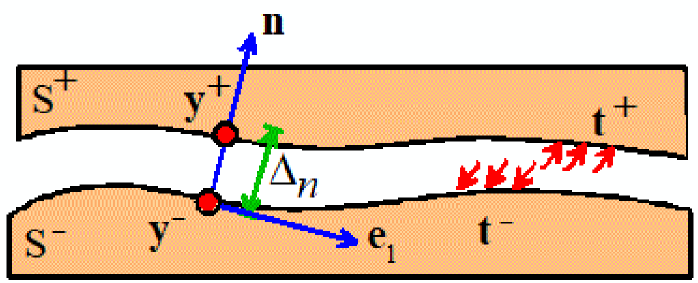
Review – Contact Geometry

$$\mathbf{y}^{+} = \mathbf{y}^{-} + \Delta_n \mathbf{n}$$

$$\mathbf{v} = \frac{d}{dt}(\mathbf{y}^{+} - \mathbf{y}^{-}) - \Delta_n \frac{d\mathbf{n}}{dt}$$

$$\mathbf{t}^{-} = T_n \mathbf{n} + T_1 \mathbf{e}_1 + T_2 \mathbf{e}_2$$



Friction Laws

Coulomb Friction

No overlap $\Delta_n \geq 0$

Contact forces must be compressive $T_n \leq 0$

No slip $\sqrt{T_1^2 + T_2^2} < \mu |T_n|$

Slip $T_1 = \mu |T_n| v_1 / \sqrt{v_1^2 + v_2^2}$ $\quad T_2 = \mu |T_n| v_2 / \sqrt{v_1^2 + v_2^2}$

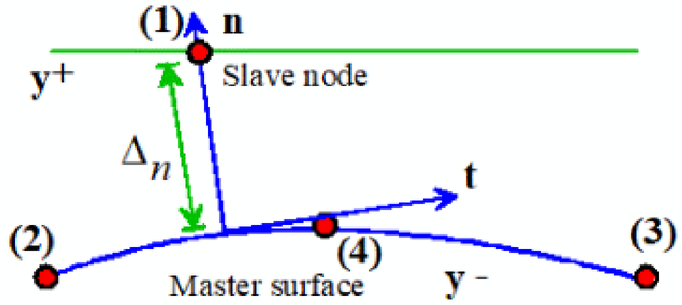
Review – FE Implementation

Consider simple frictionless contact

No overlap $\Delta_n \geq 0$

Contact forces must be compressive $T_n \leq 0$

Master surface geom $\mathbf{y} = \sum_{a=2}^{n} N^a(\xi)\mathbf{y}^a$



$N^1 = -1$

$N^2 = -\xi(1-\xi)/2 \quad N^3 = \xi(1+\xi)/2 \quad N^4 = (1-\xi^2)$

Tangent $\mathbf{t} = \sum_{a=1}^{n} \frac{dN^a}{d\xi}\mathbf{y}^a \Big/ \sqrt{\left(\sum_{a=1}^{n} \frac{dN^a}{d\xi}\mathbf{y}^a\right) \cdot \left(\sum_{a=1}^{n} \frac{dN^a}{d\xi}\mathbf{y}^a\right)}$

Normal $\mathbf{n} = \mathbf{k} \times \mathbf{t} \qquad \Delta_n \mathbf{n} = -\sum_{a=1}^{n} N^a(\xi^*)\mathbf{y}^a \Rightarrow \Delta_n = -\mathbf{n} \cdot \sum_{a=1}^{n} N^a(\xi^*)\mathbf{y}^a$

where ξ^* satisfies $\quad \mathbf{n} \cdot \mathbf{t} = 0 \Rightarrow \left(\sum_{a=1}^{n} N^a(\xi^*)\mathbf{y}^a\right) \cdot \left(\sum_{a=1}^{n} \frac{dN^a}{d\xi}\mathbf{y}^a\right) = 0$

Adding contact constraints to FE equations

We have a constraint equation $\Delta_n(u_k^b) = 0$
Enforce this with Lagrange multipliers

Augment PVW with additional terms

$$\int_R \sigma_{ij} \frac{\partial N^a}{\partial y_j} \eta_i^a + \underbrace{\delta\lambda \Delta_n + \lambda \frac{\partial \Delta_n}{\partial u_i^a} \eta_i^a}_{\text{Add these using contact elements}} - \int_{S_2} t_i^* N^a \eta_i^a = 0$$

$\forall \eta_i^a \quad \lambda$

Create "contact elements" with DOF

$$[u_i^1 \quad u_i^2 \quad u_i^3 \quad u_i^4, \quad \lambda]$$

$\Rightarrow$ Supplied to element ; return $\underline{R}^{el}$, $[k]^{el}$

Focus on calculating $\frac{\partial \Delta_n}{\partial u_i^a}$

Recall $\Delta_n \underset{\sim}{n} = -N^a \underset{\sim}{y}^a$ a is summing over element nodes

Perturb: $\delta \Delta_n \underset{\sim}{n} + \Delta_n \delta \underset{\sim}{n} = -\frac{\partial N^a}{\partial \xi} \delta \xi^* \underset{\sim}{y}^a - N^a \delta \underset{\sim}{y}^a$

($\Delta_n \delta \underset{\sim}{n}$: Parallel to $\underset{\sim}{t}$; $-\frac{\partial N^a}{\partial \xi} \delta \xi^* \underset{\sim}{y}^a$: Parallel to $\underset{\sim}{t}$)

Dot with $\underset{\sim}{n}$:

$$\delta \Delta_n = -N^a \underset{\sim}{y}^a \cdot \underset{\sim}{n}$$

Recall $\underset{\sim}{y}^a = \underset{\sim}{x}^a + \underset{\sim}{u}^a$

$$\Rightarrow \boxed{\frac{\partial \Delta_n}{\partial u_i^a} = -N^a n_i}$$

Hence residual force vector for contact element is

$$\underset{\sim}{R} = - \left[-N^1 n_1 \lambda \; -N^1 n_2 \lambda \; -N^2 n_1 \lambda \; -N^2 n_2 \lambda \cdots \quad \Delta_n \right]$$

Now focus on stiffness $-\dfrac{\partial \underset{\sim}{R}}{\partial \underset{\sim}{u}}$ and $\dfrac{\partial \underset{\sim}{R}}{\partial \lambda}$

Stiffness has form

$$[k^{el}] = \begin{bmatrix} k^{uu}_{aibk} = -\dfrac{\partial R_{ia}}{\partial u_k^b} & \begin{matrix} -N^1 n_1 \\ -N^1 n_2 \\ -N^2 n_1 \\ \vdots \end{matrix} \\ -N^1 n_1 \quad -N^1 n_2 \quad \cdots & 0 \end{bmatrix}$$

We need $\dfrac{\partial(-N^a n_i \lambda)}{\partial u_k^b} = k_{aibk}^{u}$

$$k_{aibk}^{uu} = -\frac{\partial N^a}{\partial \xi}\frac{\partial \xi^*}{\partial u_k^b} n_i \lambda - N^a \frac{\partial n_i}{\partial u_k^b}\lambda$$

Try $\dfrac{\partial \xi^*}{\partial u_k^b}$: Recall $\Delta_n \underline{n} = -N^a y^a$

Perturb $\delta\Delta_n \underline{n} + \Delta_n \delta\underline{n} = -\dfrac{\partial N^a}{\partial \xi}\delta\xi^* y^a - N^a \delta y^a$

ABAQUS assumes $\Delta_n \approx 0$

Dot with $\underline{t}$ on both sides

$$\Rightarrow \quad 0 = -\frac{\partial N^a}{\partial \xi} y^a \cdot \underline{t}\ \delta\xi^* \quad - N^a \delta y^a \cdot \underline{t}$$

Recall $\quad \underline{t} = + \frac{1}{q}\left(\frac{\partial N^c}{\partial \xi} y^c\right) \qquad q = \left| \frac{\partial N^b}{\partial \xi} y^b \right|$

$$\Rightarrow \quad \delta\xi^* = -\frac{N^a \underline{t}}{q}\ \delta y^a$$

$$\Rightarrow \quad \boxed{\frac{\partial \xi^*}{\partial u_k^b} = -\frac{N^b}{q} t_k}$$

Now consider $\dfrac{\partial n_i}{\partial u_k^b}$

Recall $\underset{\sim}{t} \cdot \underset{\sim}{n} = 0 \Rightarrow \delta \underset{\sim}{t} \cdot \underset{\sim}{n} + \underset{\sim}{t} \cdot \delta \underset{\sim}{n} = 0$

Note $\delta \underset{\sim}{n}$ is parallel to $\underset{\sim}{t}$

$\Rightarrow |\delta \underset{\sim}{n}| = - \delta \underset{\sim}{t} \cdot \underset{\sim}{n}$

Recall $\underset{\sim}{t} = \dfrac{1}{q} \dfrac{\partial N^c}{\partial \xi} y^c$

$$\Rightarrow \delta \underset{\sim}{t} = \frac{1}{q} \frac{\partial^2 N^c}{\partial \xi^2} y^c \, \delta \xi^* + \frac{1}{q} \frac{\partial N^c}{\partial \xi} \delta y^c - \frac{1}{q^2} \delta q \frac{\partial N^c}{\partial \xi} y^c$$

$$\Rightarrow \delta \underset{\sim}{t} \cdot \underset{\sim}{n} = \frac{1}{q} \frac{\partial^2 N^c}{\partial \xi^2} y^c \cdot \underset{\sim}{n} \, \delta \xi^* + \frac{1}{q} \frac{\partial N^c}{\partial \xi} \underset{\sim}{n} \cdot \delta y^c$$

Hence $|\delta \underline{n}| = -\frac{1}{q}\left\{-\frac{\partial^2 N^c}{\partial \xi^2} y^c \cdot \underline{n} \frac{1}{q} N^b \underline{t} + \frac{\partial N^b}{\partial \xi} \underline{n}\right\} \cdot \delta y^b$

$\delta \underline{n} = |\delta \underline{n}| \, \underline{t}$

$$\frac{\partial n_i}{\partial u_k^b} = -\frac{t_i}{q}\left\{\overbrace{-\frac{\partial^2 N^c}{\partial \xi^2} y^c \cdot \underline{n} \frac{1}{q}}^{p} N^b t_k + \frac{\partial N^b}{\partial \xi} n_k\right\}$$

Hence
$$k^{uu}_{aibk} = \frac{t_i}{q} N^a \left\{\frac{-p}{q} N^b t_k + \frac{\partial N^b}{\partial \xi} n_k\right\} \lambda + \frac{\partial N^a}{\partial \xi} n_i \lambda \frac{N^b t_k}{q}$$

Summary of contact element

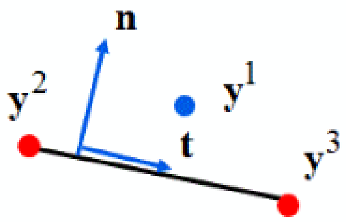
DOF: $\mathbf{u} = \begin{bmatrix} u_1^1 & u_2^1 & u_1^2 & u_2^2 & u_1^3 & u_2^3 & \lambda \end{bmatrix}$

Contact interpolation functions:

$\hat{N}^1 = -1$

$\hat{N}^2 = (1-\xi)/2$

$\hat{N}^3 = (1+\xi)/2$



Locating the contact point: $\left(\frac{\partial \hat{N}^b}{\partial \xi}\mathbf{y}^b\right)\cdot\left(\hat{N}^a\mathbf{y}^a\right) = 0$

$$\mathbf{t} = \frac{1}{q}\frac{\partial \hat{N}^a}{\partial \xi}\mathbf{y}^a \qquad q = \sqrt{\frac{\partial \hat{N}^a}{\partial \xi}\mathbf{y}^a \cdot \frac{\partial \hat{N}^c}{\partial \xi}\mathbf{y}^c}$$

$$\mathbf{n} = \mathbf{e}_3 \times \mathbf{t}$$

$$\mathbf{r} = -\begin{bmatrix} -\lambda\hat{N}^1 n_1 & -\lambda\hat{N}^1 n_2 & -\lambda\hat{N}^2 n_1 & -\lambda\hat{N}^2 n_2 & -\lambda\hat{N}^3 n_1 & -\lambda\hat{N}^3 n_2 & \Delta_n \end{bmatrix}^T$$

$$\mathbf{k} = \frac{\lambda}{q}(\boldsymbol{\mu}\otimes\boldsymbol{\tau} + \boldsymbol{\tau}\otimes\boldsymbol{\mu}) - \frac{\lambda p}{q^2}\boldsymbol{\tau}\otimes\boldsymbol{\tau} - \boldsymbol{\omega}\otimes\mathbf{v} - \mathbf{v}\otimes\boldsymbol{\omega}$$

$$q = \sqrt{\frac{\partial N^a}{\partial \xi}\mathbf{y}^a \cdot \frac{\partial N^b}{\partial \xi}\mathbf{y}^b} \qquad p = \mathbf{n}\cdot\frac{\partial^2 N^a}{\partial \xi^2}\mathbf{y}^a$$

$$\boldsymbol{\tau} = \begin{bmatrix} \hat{N}^1 t_1 & \hat{N}^1 t_2 & \hat{N}^2 t_1 & \hat{N}^2 t_2 & \hat{N}^3 t_1 & \hat{N}^3 t_2 & 0 \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} \hat{N}^1 n_1 & \hat{N}^1 n_2 & \hat{N}^2 n_1 & \hat{N}^2 n_2 & \hat{N}^3 n_1 & \hat{N}^3 n_2 & 0 \end{bmatrix}$$

$$\boldsymbol{\mu} = \begin{bmatrix} \frac{\partial \hat{N}^1}{\partial \xi}n_1 & \frac{\partial \hat{N}^1}{\partial \xi}n_2 & \frac{\partial \hat{N}^2}{\partial \xi}n_1 & \frac{\partial \hat{N}^2}{\partial \xi}n_2 & \frac{\partial \hat{N}^3}{\partial \xi}n_1 & \frac{\partial \hat{N}^3}{\partial \xi}n_2 & 0 \end{bmatrix}$$

$$\boldsymbol{\omega} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$$

Finally we need an algorithm to decide whether slave nodes are in contact with master

$$\Delta_n \geq 0 \qquad \lambda \leq 0$$

Iteration

(1) Guess set of slave nodes with $\Delta_n = 0$
(Start with empty set; or use previous increment

(2) Solve equilibrium (Newton-Raphson)

(3) Count: # slave nodes w $\lambda \geq 0$ i
slave nodes w $\Delta_n \leq 0$ j
slaves that slide to new master k

(4) $i+j+k > 0$ update set of constrained slaves

(5) Next step