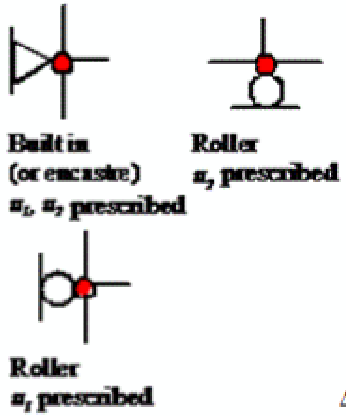
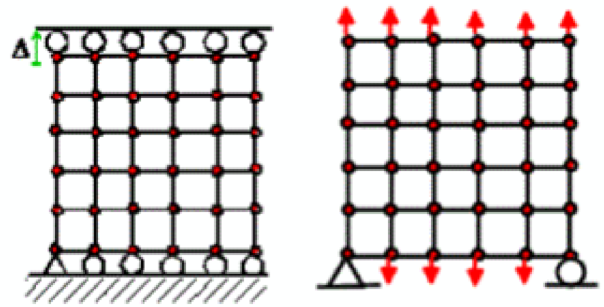
Review

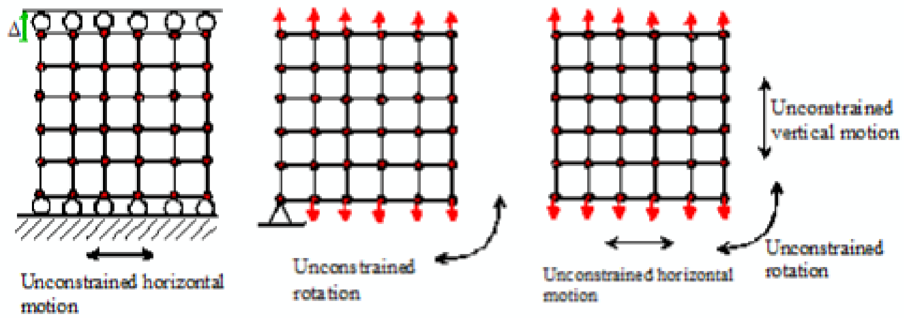
Boundary Conditions

We can apply
1. Prescribed displacements
2. Forces on nodes
3. Pressure on element faces
4. Body forces
5. For some elements, can apply rotations/moments



Properly constrained solids





Incorrectly constrained solids

Constraints Equations that relate DOF

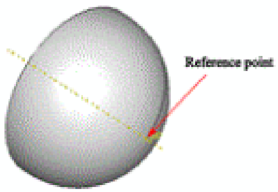
Shell elements connected to solid
We would need to connect the shell to the solid with a constraint

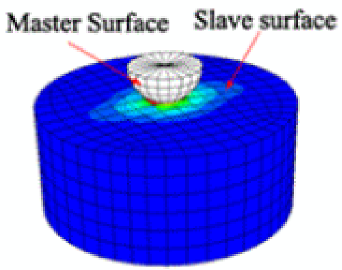
Review

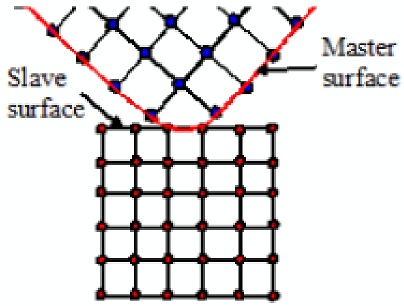
Contact

Master/slave pairs

Select
1. Contact algorithm
2. Constitutive law for contact (eg friction)
3. "Soft" or "Hard" contact



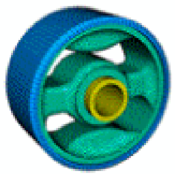


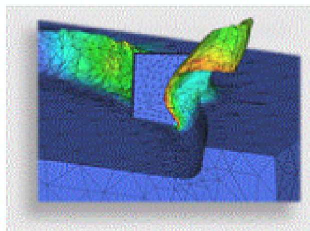


Nodes on slave surface are prevented from penetrating inside master surface

Solution Procedures

Small strain –v- large strain (NLGEOM)





Static analysis
Solves $\mathbf{R}(\mathbf{u}) = \mathbf{F}^*$ $\quad \mathbf{u} = \mathbf{u}^*$ using Newton-Raphson iteration

Explicit Dynamics: solves $\mathbf{M}\ddot{\mathbf{u}} + \mathbf{R}(\mathbf{u}) = \mathbf{F}^*$ $\quad \mathbf{u} = \mathbf{u}^*$ using 2nd order forward Euler scheme

Implicit Dynamics: solves $\mathbf{M}\ddot{\mathbf{u}} + \mathbf{R}(\mathbf{u}) = \mathbf{F}^*$ $\quad \mathbf{u} = \mathbf{u}^*$ using 2nd order backward Euler scheme

Special procedures: modal dynamics, buckling ('Linear Perturbation steps')

Topics for todays class

- Implicit dynamics
- Units in FEA
- Using dimensionless variables in simulations; scaling governing equations
- Brief look at homework 1
- FEA for static linear elasticity
  - Governing equations for elastostatics
  - The principle of minimum potential energy; estimating displacement fields by minimizing energy
  - Simple FEA program for linear elasticity – 2D plane strain with constant strain triangular elements

2.6.2 Implicit dynamics

- Solves $\underline{F} = m\underline{a}$ using backward - Euler :

(a) $\underline{v}(t+\Delta t) = \underline{v}(t) + \underline{a}\,\Delta t$

(b) $\underline{u}(t+\Delta t) = \underline{u}(t) + \underline{v}(t+\Delta t)\,\Delta t + \underline{a}\,\Delta t^2/2$

(c) $\underline{a} = [M]^{-1}\left(\underline{F}^* - R\left(\underline{u}(t+\Delta t)\right)\right)$

Solve (a - c) for $\underline{u}(t+\Delta t)$

Unconditionally stable - Δt can be chosen freely

- Does not conserve energy for large Δt

- Δt may need to be small for accuracy

2.7 Units and scaling in FEA

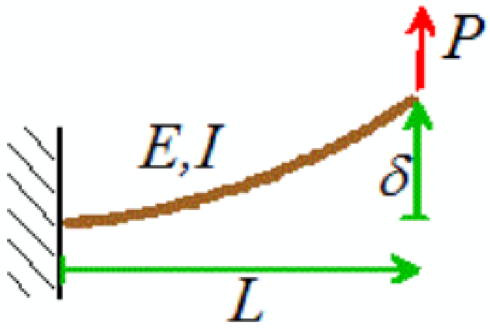
FEA is solving $F = ma$ – no units a-priori
- any consistent set is fine

Be careful with length – if we sketch a part in mm or cm, that sets length unit

If force is in N, stress is in N/mm^2 or N/cm^2

2.7.1 Using scaling / dimensional analysis

Example: Cantilever beam

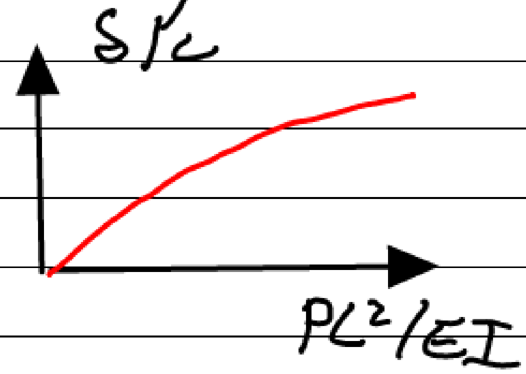


$$\delta = f(P, E, I, L)$$ ← 4 variables

$$\frac{\delta}{L} = g\left(\frac{P}{EL^2}, \frac{I}{L^4}\right)$$ ← 2 variables

If we know solution only depends on product EI then can simplify furthe.

$$\delta / L = h\left(PL^2 / EI \right)$$



If we solved a linear problem δ & P

$$\Rightarrow \quad \delta / L = \beta \frac{PL^2}{EI} \qquad \beta \text{ is a constant}$$

- now just need 1 FEA simulation

This approach works but a better method is to scale governing equations

For beam problem we solve for deflection $w(x)$

- Eqs: $EI\, w^{IV} = 0$
- BCs: $\left.\begin{array}{l} w = 0 \\ w' = 0 \end{array}\right\} x = 0$ $\qquad \left.\begin{array}{l} w'' = 0 \\ EI\, w^{III} = P \end{array}\right\} x = L$

Let $w = L\,\hat{w}(\xi)$ $\quad x = \xi L$ $\quad 0 < \xi < 1$ $\quad \dfrac{\partial}{\partial x} = \dfrac{1}{L}\dfrac{\partial}{\partial \xi}$

Governing eqs are now

$$\cancel{\frac{EI}{L^3}}\ \frac{\partial^4 \hat{w}}{\partial \xi^4} = 0$$

$$\left.\begin{array}{l} \hat{w} = 0 \\ \dfrac{\partial \hat{w}}{\partial \xi} = 0 \end{array}\right\} \xi = 0 \qquad \left.\begin{array}{l} \partial^2 \hat{w}/\partial \xi^2 = 0 \\ \dfrac{EI}{L^2}\dfrac{\partial^3 \hat{w}}{\partial \xi^3} = P \end{array}\right\} \xi = 1$$

Combine

Hence solution depends on $\frac{PL^2}{EI}$

We can simplify further by defining

$$w = \frac{EI}{PL^2}\hat{W}$$

Governing eqs $\quad \partial^4 w / \partial \xi^4 = 0$

BCs $\quad \left.\begin{array}{c} \partial w / \partial \xi = 0 \\ w = 0 \end{array}\right\} \xi = 0 \qquad \left.\begin{array}{c} \partial^2 w / \partial \xi^2 = 0 \\ \frac{\partial^3 w}{\partial \xi^3} = 1 \end{array}\right\} \xi = 1$

w applies to all beams!

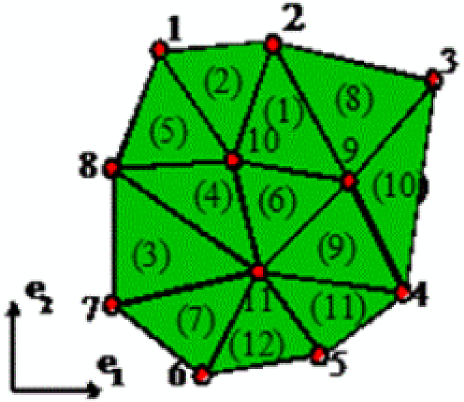
To find $\quad \delta = \frac{PL^3}{EI} w(\xi = 1)$

Preview – Basic FEA for static linear elasticity

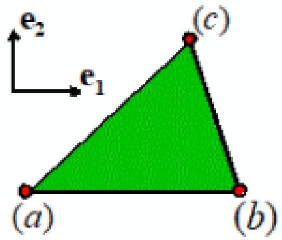
Background: Governing equations for linear elasticity
The Principle of Minimum Potential Energy

- Plane strain linear elasticity problem as an energy minimization

$$\Phi = \int_A \phi dA - \int_{S_2} \mathbf{t}^* \cdot \mathbf{u} ds$$



- Interpolating the displacement field using constant strain triangles
- Computing the potential energy
  - Strains, stresses and strain energy density inside an element. Element stiffness matrix
  - Potential energy of a loaded element face – element force vector
  - Total potential energy – global stiffness and global force
  - Constrained boundary conditions

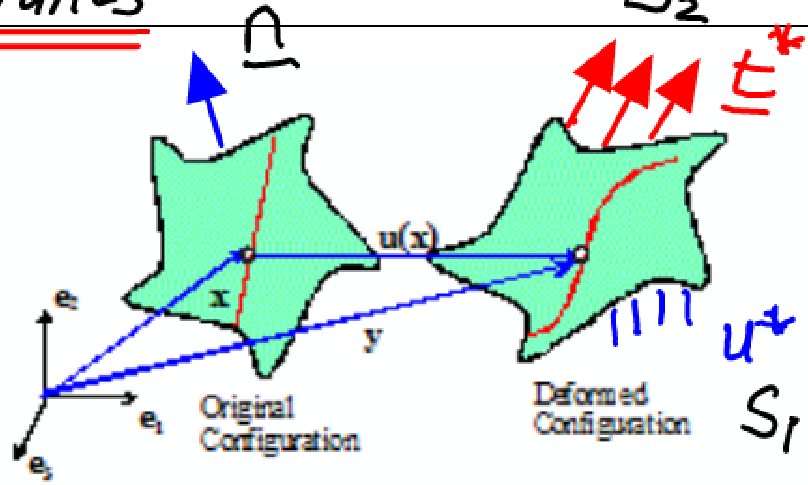


- Minimizing the total potential energy

$$\Phi = \frac{1}{2}\left(\mathbf{u}^{Global}\right)^T [\mathbf{K}]\mathbf{u}^{Global} - \left(\mathbf{u}^{Global}\right)^T \cdot \mathbf{f}^{Global} \quad \Rightarrow \quad [K]\mathbf{u}^{Global} = \mathbf{f}^{Global}$$

- Implementation
  - Data structures for mesh and BC definition
  - Assembling the element and global stiffness matrices
  - Prescribing boundary conditions
  - Solution and post-processing

3 FEA for plane linear elastostatics



Background: Governing Equations

Find : $[u_i, \varepsilon_{ij}, \sigma_{ij}]$

Satisfying

① $$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

② $$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$$

$$C_{ijkl} = \frac{E}{2(1+\nu)}(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) + \frac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl}$$

③ $$\frac{\partial \sigma_{ij}}{\partial x_i} = 0 \qquad \sigma_{ij} = \sigma_{ji}$$

E - Youngs modulus
ν - Poissons ratio

④ $u_i = u_i^*$ on S_1 $\qquad \sigma_{ij}\, n_i = t_j^*$ on S_2

Principle of minimum potential energy

Let v_i be a differentiable vector field on R

Let $\hat{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$

Strain energy density $\phi = \frac{1}{2} C_{ijkl} \hat{\varepsilon}_{ij} \hat{\varepsilon}_{kl}$

Define total PE $\quad \underline{\Phi} = \int_V \phi \, dV - \int_{S_2} t_i^* v_i \, dA$

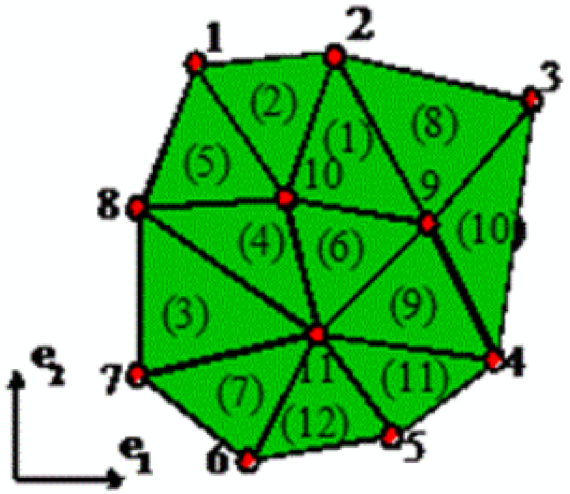
Then $\quad \underline{\Phi}(\underline{v}) \geq \underline{\Phi}(\underline{u})$

and $\quad \underline{\Phi}(\underline{v}) = \underline{\Phi}(\underline{u}) \iff \underline{v} = \underline{u}$

Application: Approximate $\underline{u}$ in some way. Get best approximation by minimizing $\bar{\Phi}$

3.1) Plane linear elastostatic FE with constant strain triangular elements

(a) Divide solid into triangles



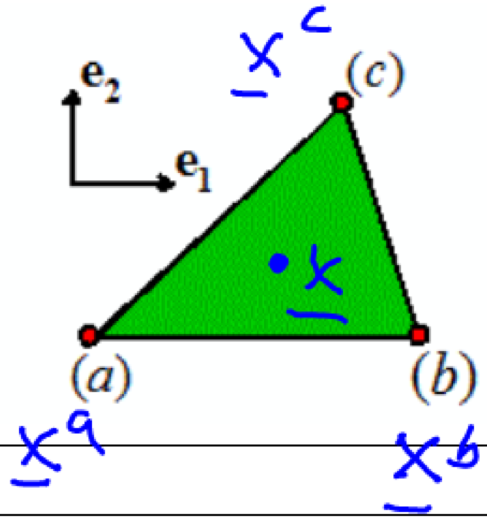
(b) Let u_i^a denote unknown displacement @ nodes $a = 1 \ldots N$

(c) Interpolate $\underline{u}$ inside each triangle eg for a point inside triangle with vertices a, b, c

$$\underline{u} = N^a(\underline{x})\, \underline{u}^a + N^b(\underline{x})\, \underline{u}^b + N^c(\underline{x})\, \underline{u}^c$$

(d) Find $\bar{\Phi}$; minimize wrt $\underline{u}^a$

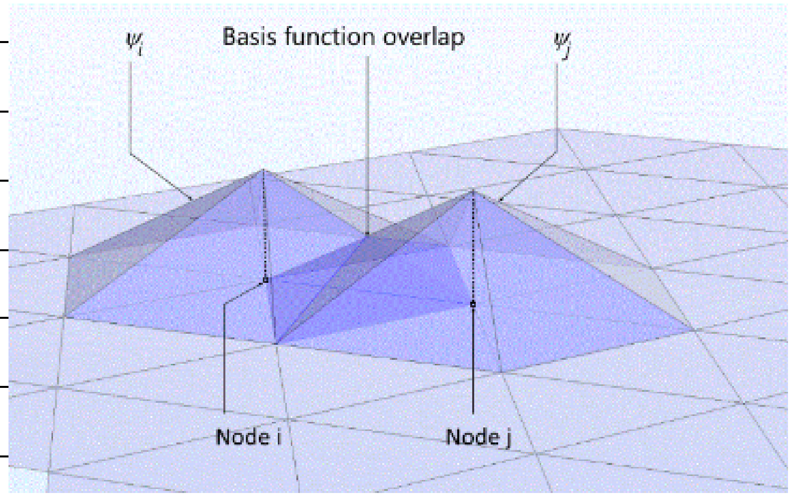
- Interpolation functions for constant strain triangle



$$N^a(x_1,x_2)=\frac{\left(x_2-x_2^{(b)}\right)\left(x_1^{(c)}-x_1^{(b)}\right)-\left(x_1-x_1^{(b)}\right)\left(x_2^{(c)}-x_2^{(b)}\right)}{\left(x_2^{(a)}-x_2^{(b)}\right)\left(x_1^{(c)}-x_1^{(b)}\right)-\left(x_1^{(a)}-x_1^{(b)}\right)\left(x_2^{(c)}-x_2^{(b)}\right)}$$

$$N^b(x_1,x_2)=\frac{\left(x_2-x_2^{(c)}\right)\left(x_1^{(a)}-x_1^{(c)}\right)-\left(x_1-x_1^{(c)}\right)\left(x_2^{(a)}-x_2^{(c)}\right)}{\left(x_2^{(b)}-x_2^{(c)}\right)\left(x_1^{(a)}-x_1^{(c)}\right)-\left(x_1^{(b)}-x_1^{(c)}\right)\left(x_2^{(a)}-x_2^{(c)}\right)}$$

$$N^c(x_1,x_2)=\frac{\left(x_2-x_2^{(a)}\right)\left(x_1^{(b)}-x_1^{(a)}\right)-\left(x_1-x_1^{(a)}\right)\left(x_2^{(b)}-x_2^{(a)}\right)}{\left(x_2^{(c)}-x_2^{(a)}\right)\left(x_1^{(b)}-x_1^{(a)}\right)-\left(x_1^{(c)}-x_1^{(a)}\right)\left(x_2^{(b)}-x_2^{(a)}\right)}$$



Notice $N^a = 1$ $N^b = N^c = 0$ @ $\underline{x} = \underline{x}^a$

etc

Interpolations are linear functions of position