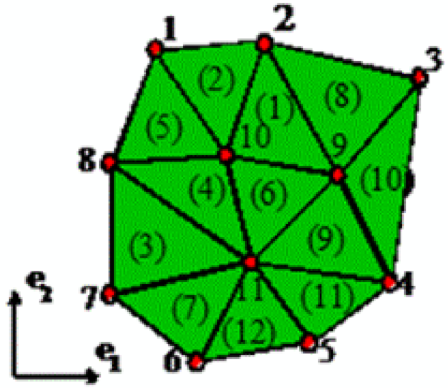
Review: FEA as a Galerkin Method

- Strong form of equilibrium equation (small strains)

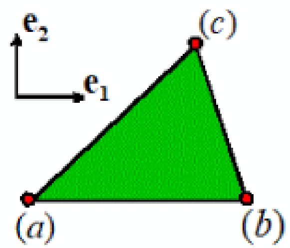
$$\frac{\partial}{\partial x_j}\left(C_{ijkl}\frac{\partial u_k}{\partial x_l}\right) + b_i = 0$$

$$t_i^* - C_{ijkl}\frac{\partial u_k}{\partial x_l} n_j = 0$$



- Introduce test function η_i satisfying $\eta_i = 0 \;\; on \;\; S_1$ multiply strong form and integrate

$$\int_R \frac{\partial}{\partial x_j}\left(C_{ijkl}\frac{\partial u_k}{\partial x_l}\right)\eta_i dV_0 + \int_R b_i \eta_i dV_0 + \int_{\partial_2 R}\left(t_i^* - C_{ijkl}\frac{\partial u_k}{\partial x_l} n_j\right) dA = 0 \qquad \forall \; admiss \; \eta_i$$



- Integrate first term by parts

$$-\int_V C_{ijkl}\frac{\partial u_k}{\partial x_l}\frac{\partial \eta_i}{\partial x_j} dV_0 + \int_V b_i \eta_i dV_0 + \int_{S_2} t_i^* \eta_i dA = 0 \qquad \forall \; admiss \; \eta_i$$

- Insert interpolation $\quad u_i(\mathbf{x}) = \sum_{a=1}^{n} N^a(\mathbf{x}) \delta u_i^a \qquad \eta_i(\mathbf{x}) = \sum_{a=1}^{n} N^a(\mathbf{x}) \eta_i^a$

$$\left(K_{aibk} u_k^b - F_i^a \right)\eta_i^a = 0 \Rightarrow K_{aibk} u_k^b = F_i^a$$

$$K_{aibk} = \int_{\Re} C_{ijkl}\frac{\partial N^a(\mathbf{x})}{\partial x_j}\frac{\partial N^b(\mathbf{x})}{\partial x_l} dV \qquad F_i^a = \int_R b_i N^a(\mathbf{x}) dV + \int_{S_2} t_i^* N^a(\mathbf{x}) dA$$

Topics for todays class

- Perspectives on FEA for linear elasticity
  - FEA as a best approximation method
  - Conditions for FEA convergence
- Problems with standard elements
  - 'Shear Locking' in beam/plate/shell problems
  - 'Volumetric locking' in near-incompressible materials

5.2 FEA as a "best approximation" method

Linear Elasticity; elastic constants C_{ijkl}; t_i^* prescribed on S_2

Let $\underline{u}$ be exact solution

$\underline{u}^h = \sum_a N^a \underline{u}^a$ be the FEA solution

Define $\mathcal{E}(\underline{u} - \underline{u}^h)$ as error measure

$$\mathcal{E}(\underline{u} - u_h) = \underbrace{\int_R \frac{1}{2} C_{ijkl} \frac{\partial}{\partial x_l} (u_k - u_k^h) \frac{\partial}{\partial x_j} (u_i - u_i^h)\, dV}_{\text{Strain energy of } \underline{u} - \underline{u}^h}$$

Then FEA minimizes $\mathcal{E}$ wrt $\underline{u}^a$

Proof: Re-write $\mathcal{E}$ as

$$\mathcal{E}(\underline{u}) + \mathcal{E}(\underline{u}^h) - \int_R \frac{1}{2} \underbrace{\left(C_{ijkl} + C_{klij} \right)}_{\text{Equal}} \frac{\partial u_k}{\partial x_l} \frac{\partial u_i^h}{\partial x_j}$$

Note: $C_{ijkl} = C_{klij}$

$$C_{ijkl} \frac{\partial u_k}{\partial x_l} = \sigma_{ij}$$

$$\int_R \sigma_{ij} \frac{\partial u_i^h}{\partial x_j} = \int_R \frac{\partial}{\partial x_j} \sigma_{ij} u_i^h - \cancelto{0}{\frac{\partial \sigma_{ij}}{\partial x_j}} u_i^h$$

$$= \int_S \sigma_{ij} u_i^h n_j \, dA = \int_{S_2} t_i^* u_i^h \, dA + \int_{S_1} \sigma_{ij} n_j u_i^h \, dA$$

$$\mathcal{E}(\underset{\sim}{u} - \underset{\sim}{u}^h) = \mathcal{E}(u) + \underbrace{\mathcal{E}(\underset{\sim}{u}^h) - \int_{S_2} t_i^* u_i^h \, dA}_{\text{Potential energy of FEA solution}} - \int_{S_1} \sigma_{ij} n_j u_i^h \, dA$$

Now minimize $\mathcal{E}$ wrt $\underset{\sim}{u}^h$, with fixed $\underset{\sim}{u}^h = u^*$ on S_1

$\Rightarrow$ Minimize PE of $\underset{\sim}{u}^h$

$\Rightarrow$ FEA equations follow!

5.3 Conditions for convergence of FEA for linear elasticity

State results w/o proof: see Bathe chap 4 for derivations

Factors affecting convergence

① m : Order of highest derivative appearing in weak form of PDE

Examples 2D/3D linear elasticity

$$\int C_{ijkl} \frac{\partial u_k}{\partial x_l} \frac{\partial \eta_i}{\partial x_j} \, dV - \int t_i^* \eta_i \, dA \Rightarrow m = 1$$

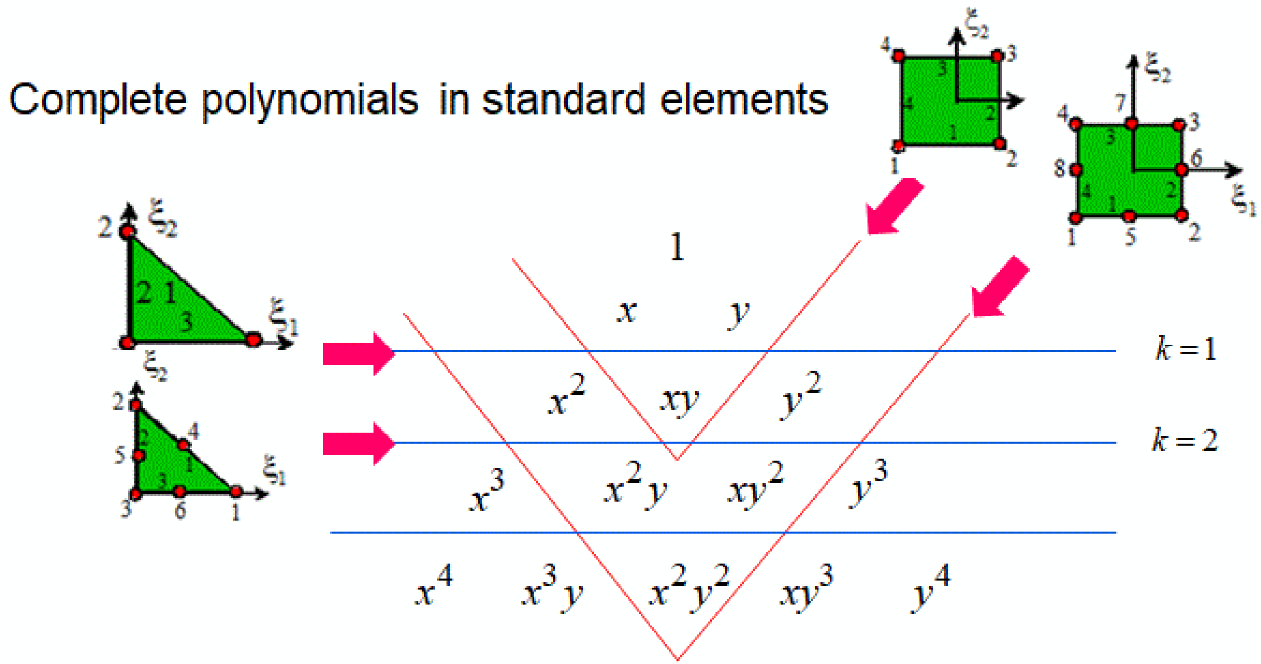
Beam: $\int_0^L \left(EI \frac{d^2 w}{dx^2} \frac{d^2 \eta}{dx^2} - q\,\eta \right) dx \Rightarrow m = 2$

② : k : Order of highest complete polynomial in interpolation functions.

k is often displayed as Pascal triangle for 2D elements

$k=1$ for linear

$k=2$ for quadratic



③ Element size and shape

Let h_e be smallest sphere enclosing element
ρ_e " largest sphere that fits in element

Define $h = \max\{h_e\}$ $\rho = \min(\rho_e)$

Let $\sigma = \frac{h}{\rho}$

We can show that

$$\varepsilon = B(\underline{u}) \frac{h^{k+1}}{\rho^m} = B(\underline{u})\, \sigma^m\, h^{k-m+1}$$

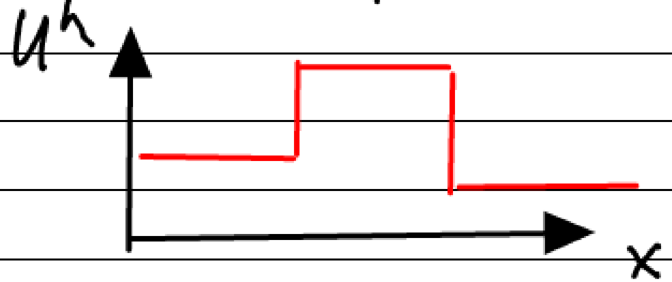
We need $\varepsilon \to 0$ as $h \to 0$

This requires:

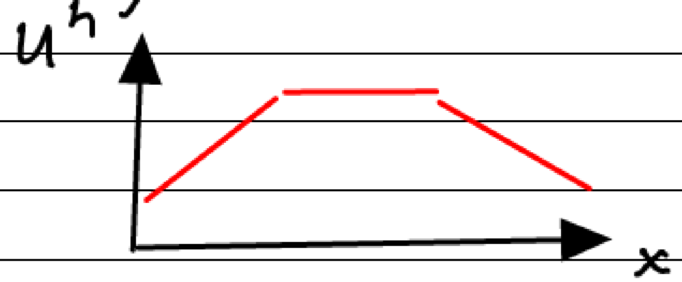
(1) $\mathcal{E}(\underline{u}^h)$ must be bounded
(2) $k - m + 1 > 0$

Consequences:

(1) $\Rightarrow$ for linear elasticity we need C^1 continuity in $\mathbb{R}$; at linear interpolation in Ω_e



$C^0 \Rightarrow$ No good



C^1 : OK

For beams: at least quadratic interpolation
Continuous slope across element boundaries

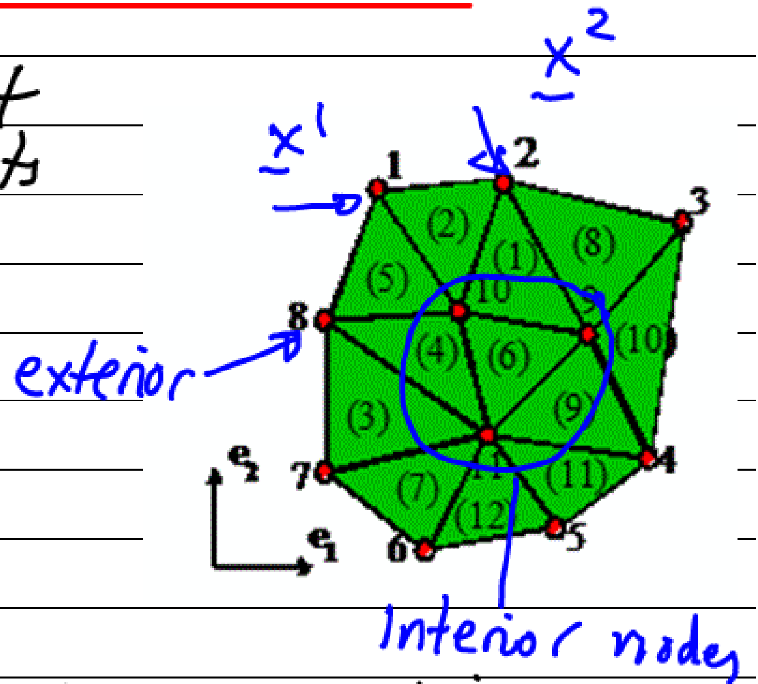
2^{nd} condition $\quad k - m + 1 > 0$

For linear elasticity $m = 1 \Rightarrow k > 0$
(at least linear)

For beam: $m = 2 \Rightarrow k > 1$; at least quadratic.

5.4 "Patch Test" - an engineering test of a solid mechanics element

Although passing patch test does not guarantee convergence, usually elements that pass are OK, others are not



Two version

① : Generate meshes (test several) of irregular elements

Apply uniform strain field to mesh by constraining nodal displacements

$$u_i^a = E_{ij} \, x_j^a \qquad (1)$$

E_{ij} ← constant strain

Patch test passed if $\underline{K}\,\underline{u} = \underline{0}$

for all interior nodes

(2) Apply (1) to all exterior nodes

Solve $[K]\underline{u} = \underline{r}$

At internal nodes we should get

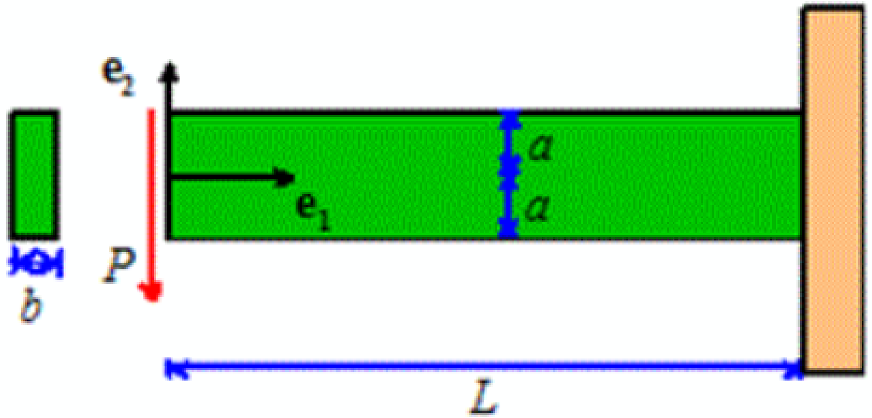
$$u_i^a = E_{ij} x_j^a$$

6) Advanced element formulations

Standard elements fail in two situations

① "Shear locking" in beams
② "Volumetric locking" in near incompressible materials

6.1 Shear locking



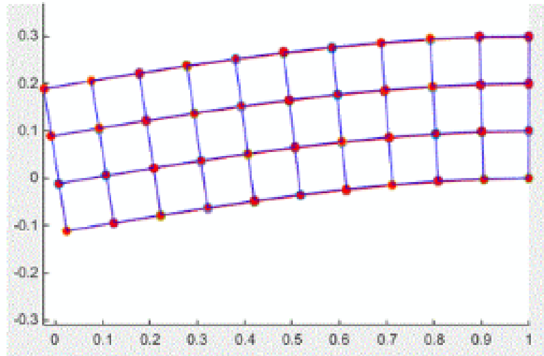
Analytical solution to 2D elasticity problem
(See Chapter 5 of solidmechanics.org for details)

$$u_1 = \frac{3P}{4Ea^3b}x_1^2x_2 - \frac{P}{4Ea^3b}(2+\nu)x_2^3 + \frac{3P}{2Ea^3b}(1+\nu)a^2x_2 - \frac{3PL^2x_2}{4Ea^3b}$$

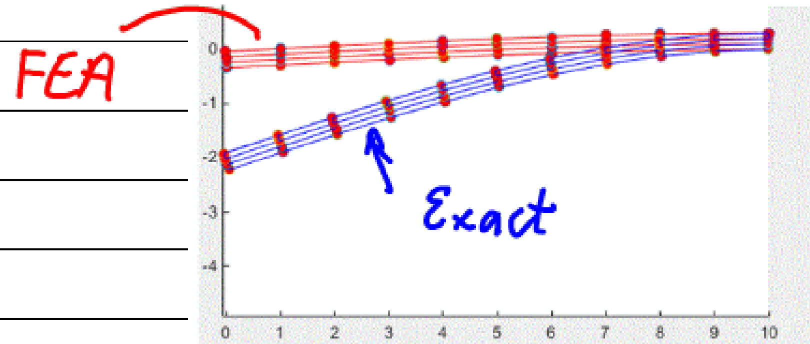
$$u_2 = -\nu\frac{3P}{4Ea^3b}x_1x_2^2 - \frac{P}{4Ea^3b}x_1^3 + \frac{3PL^2x_1}{4Ea^3b} - \frac{PL^3}{2Ea^3b}$$

Try to solve this with FEA

① Solve a short beam with linear elements
⇒ OK



② Long beam with linear elements
FEA underestimates solution by factor of 10



③ Quadratic 8 noded elements are ok