ENGN2340 Final Project:
Implementation of a Euler-Bernuolli Beam Element
Michael Monn

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Problem Definition and Shape Functions

Although there exist many analytical solutions to the Euler-Bernuolli beam equations for simple geometries and loading scenarios, complex geometries must be solved numerically. In the following sections, I derive the equations necessary for implementing an Euler-Bernuolli beam element. I then implement the discretized equations in MatLab in order to compare the finite element solution to the analytical result for several simple problems. The finite element code is then used to calculate the deformation of a simply-supported parabolic arch with a point load at its crest.

Finally, I derive the finite element equations for an Euler-Bernuolli beam that is modified to account for finite deformations due to large rotations. In this case the strains are still assumed to be small, but the problem is geometrically nonlinear. The finite deformation model is implemented in MatLab and used to verify that for small loads, the finite deformation and linear model produce the same result. A two-node planar beam element has 4 degrees of freedom, which are defined as

$$\mathbf{u} = \begin{bmatrix} u_1 & \theta_1 & u_2 & \theta_2 \end{bmatrix}$$

where u_i represent transverse nodal displacements and $\theta_i = \frac{du_i}{dx}$ represents the slope of the beam at each node. We will see later that we must be able to take second derivatives of the shape functions used for interpolating nodal values, therefore we express the displacements in terms of a cubic polynomial in order for the degrees of freedom to be continuous across elements.

$$u = c_0 + c_1 x + c2x^2 + c_3 x^3$$
$$\theta = \frac{du}{dx} = c_1 + 2c_2 x + 3c_3 x^2$$

at the nodal positions x_1 and x_2, the displacements and angles take the nodal values u_i and θ_i. Applying these boundary conditions to the form of u and θ to solve for c_i yields the shape functions $N^{(i)}$ in terms of the global coordinates

$$
\begin{aligned}
N^1 &= -(x-x_2)^2(2(x_1-x)-L)/L^3 \\
N^2 &= (x-x_1)(x-x_2)^2/L^2 \\
N^3 &= (x-x_1)^2(L+2(x_2-x))/L^3 \\
N^4 &= (x-x_1)^2(x-x_2)/L^2
\end{aligned}
$$

From here we map the element to a set of dimensionless local coordinates that vary from -1 to 1

$$\xi = \frac{2x}{L} - 1$$

This allows us to rewrite the shape functions in terms of ξ as

$$
\begin{aligned}
N^1 &= \frac{1}{4}(1-\xi)^2(2+\xi) \\
N^2 &= \frac{1}{8}L(1-\xi)^2(1+\xi) \\
N^3 &= \frac{1}{4}(1+\xi)^2(2-\xi) \\
N^4 &= \frac{-1}{8}L(1+\xi)^2(1-\xi)
\end{aligned}
$$

In order to calculate the derivatives of the shape functions with respect to the global coordinates we simply apply chain rule, noting that $\frac{d\xi}{dx} = \frac{2}{L}$.

$$
\begin{aligned}
\frac{dN^1}{dx} &= \frac{d\xi}{dx}\frac{dN^1}{dxi} = \frac{3(\xi^2-1)}{2L} \\
\frac{dN^2}{dx} &= \frac{d\xi}{dx}\frac{dN^2}{dxi} = \frac{1}{4}(\xi-1)(1+3\xi) \\
\frac{dN^3}{dx} &= \frac{d\xi}{dx}\frac{dN^3}{dxi} = \frac{-3(\xi^2-1)}{2L} \\
\frac{dN^4}{dx} &= \frac{d\xi}{dx}\frac{dN^4}{dxi} = \frac{1}{4}(\xi+1)(3\xi-1)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d^2N^1}{dx^2} &= (\frac{d\xi}{dx})^2\frac{d^2N^1}{dxi^2} = \frac{6\xi}{L^2} \\
\frac{d^2N^2}{dx^2} &= (\frac{d\xi}{dx})^2\frac{d^2N^2}{dxi^2} = \frac{3\xi-1}{L} \\
\frac{d^2N^3}{dx^2} &= (\frac{d\xi}{dx})^2\frac{d^2N^3}{dxi^2} = \frac{-6\xi}{L^2} \\
\frac{d^2N^4}{dx^2} &= (\frac{d\xi}{dx})^2\frac{d^2N^4}{dxi^2} = \frac{3\xi+1}{L}
\end{aligned}
$$

The integration points and weights used are $\xi^I = \begin{bmatrix}-c & c\end{bmatrix}$ and $w^I = \begin{bmatrix}1 & 1\end{bmatrix}$, where $c = 0.5773502692$.

Small Deformation FE Equations

The equilibrium equation for a small deformation (classical) Euler beam in strong form is

$$
EI\frac{d^4u}{dx^4} + q(x) = 0
$$

With boundary conditions $M^* = EI\frac{d^2u}{dx^2}$ and $V^* = -EI\frac{d^3u}{dx^3}$. We introduce a test function η and integrate to get

$$
\int_L EI\frac{d^4u}{dx^4}\eta dx + \int_L q(x)\eta dx - (V^* + EI\frac{d^3u}{dx^3})\eta\Big|_L - (M^* - EI\frac{d^2u}{dx^2})\frac{d\eta}{dx}\Big|_L = 0
$$

Integrating by parts, this reduces to

$$
\int_L EI\frac{d^2u}{dx^2}\frac{d^2\eta}{dx^2}dx + \int_L q(x)\eta dx - V^*\eta\Big|_L - M^*\frac{d\eta}{dx}\Big|_L = 0
$$

We now introduce an interpolation scheme (Galerkin type) using the previously derived shape functions

$$
\begin{aligned}
\eta &= N^a\eta^a \\
\frac{d\eta}{dx} &= \frac{dN^a}{dx}\eta^a \\
\frac{d^2\eta}{dx^2} &= \frac{d^2N^a}{dx^2}\eta^a \\
\frac{d^2u}{dx^2} &= \frac{d^2N^a}{dx^2}u^a
\end{aligned}
$$

We write the discretized weak form of the equilibrium equations as

$$
\left[\int_L EIu^a\frac{d^2N^a}{dx^2}\frac{d^2N^b}{dx^2}dx + \int_L q(x)N^a dx - V^*N^a\Big|_L - M^*\frac{dN^a}{dx}\Big|_L\right]\eta^a = 0 \quad \forall\eta^a
$$

Converting the integrals to the non dimensional local coordinates, we define a stiffness matrix, K_{ab}, residual vector, R_a, and load vector, F_a, as

$$K_{ab} = \frac{L}{2}\int_{-1}^{1} EI\frac{d^2N^a}{d\xi^2}\frac{d^2N^b}{d\xi^2}d\xi$$
$$R_a = \frac{L}{2}\int_{-1}^{1} qN^a d\xi$$
$$F_a = V^*N^a\Big|_{-1}^{1} + M^*\frac{dN^a}{d\xi}\Big|_{-1}^{1}$$

The degrees of freedom are found by solving the linear equation

$$u^a = K_{ab}^{-1}(F_a - R_a)$$

Comparison to Analytical Solutions

The linear EB FE code was first tested on two simple problems

1. A cantilever with a constant distributed transverse load (Figure 1).
2. A simply supported beam with a point moment applied to the right end (Figure 2).

These two example problems encompass the full range of loading options (point, and distributed) as well as both types of boundary condition (encastre and simple support). It is clear from these tests that the implementation of the FE beam element provides a very accurate approximation to the analytical solutions.

From here, we also demonstrate applicability to problems with greater geometric complexity via the deformation of a parabolic arch. Figure 3 shows the deformation of a simply supported arch due to a distributed load on the center element, while Figure 4 shows the asymmetric deformation of an arch with an off-center load and an encastered left end.

While these two examples show the power of FEA in analyzing complex geometries, they also demonstrate the limitations linear beam theory. Specifically, instabilities will not be present in the deformations derived from geometrical linear beam theory. In Figure 3, the arch is shown to deform uniformly downwards with no visible local bending near the loaded element and no evidence of snap-through buckling even at higher loads. In order to capture events like snap-through buckling or local instabilities, we must extend our analysis to include large rotations.

FE Equations Modified to Account for Large Rotations

For a slender uniform beam subjected to transverse loading (negligible axial deformation), the equilibrium equation can be extended to include large rotations as discussed in *Nonlinear Finite Element Analysis* (Reddy). In this case the strains are still considered to be small but geometric nonlinearities are accounted for.

$$EI\frac{d^4u}{dx^4} - \frac{3}{2}EA\left(\frac{du}{dx}\right)^2\frac{d^2u}{dx^2} - q(x) = 0$$

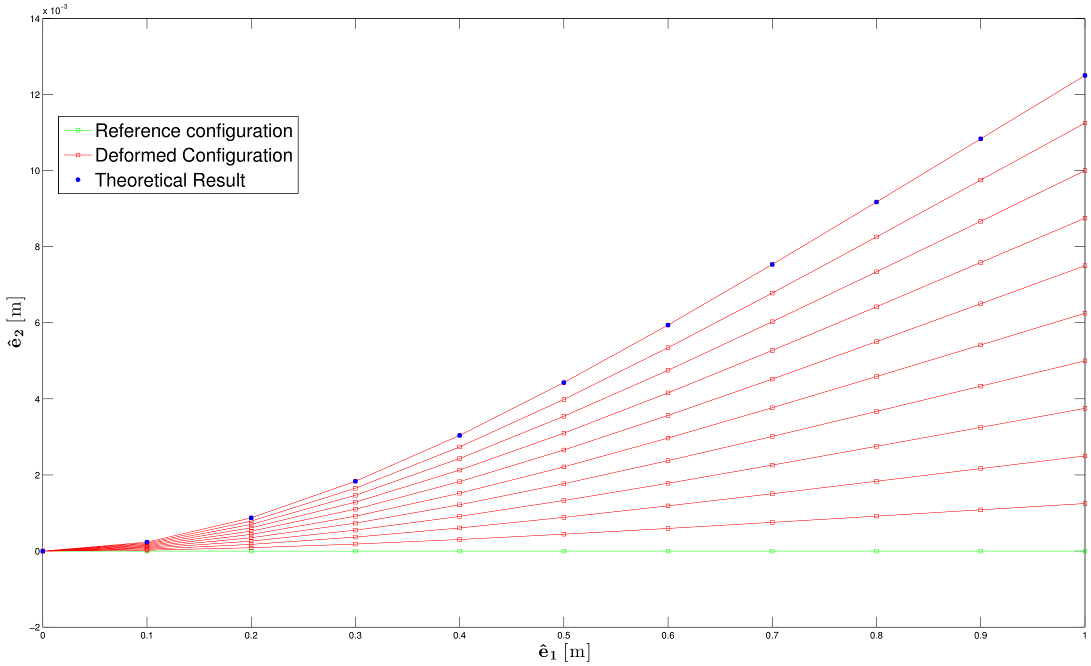

Figure 1: Finite element solution to a 10-element cantilever of unit length and $EI = 1$ with a distributed load $q = 0.1$ applied in 10 steps.

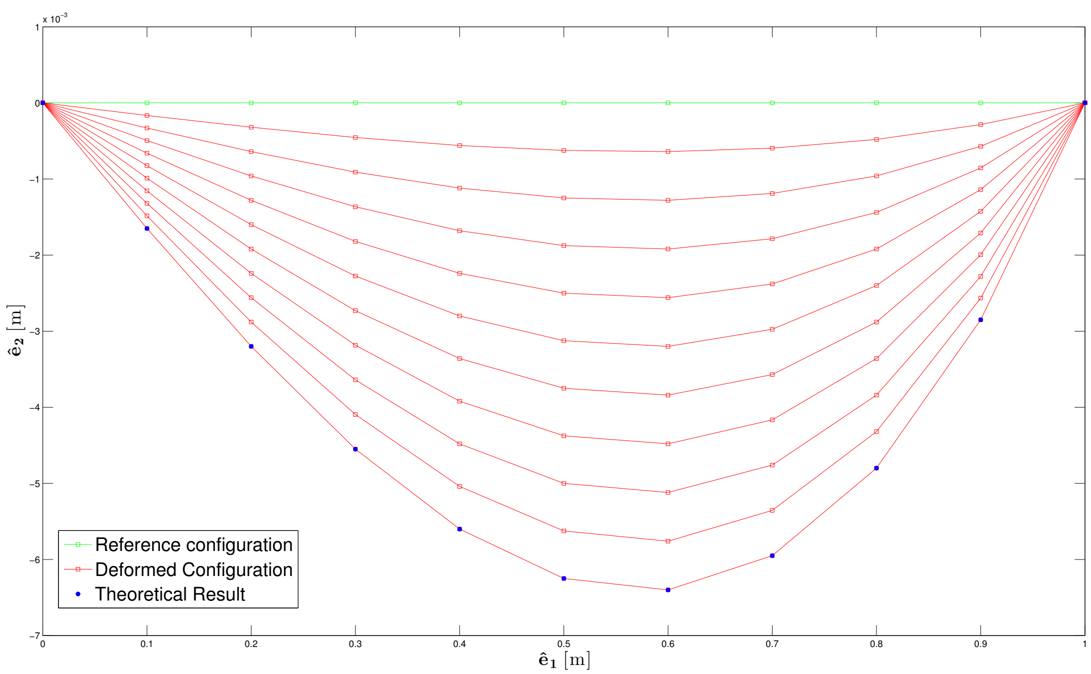

Figure 2: Finite element solution to a 10-element simply-supported beam of unit length and $EI = 1$ with a point moment applied to the last (on the right) node of $M^* = 0.1$ applied in 10 steps.

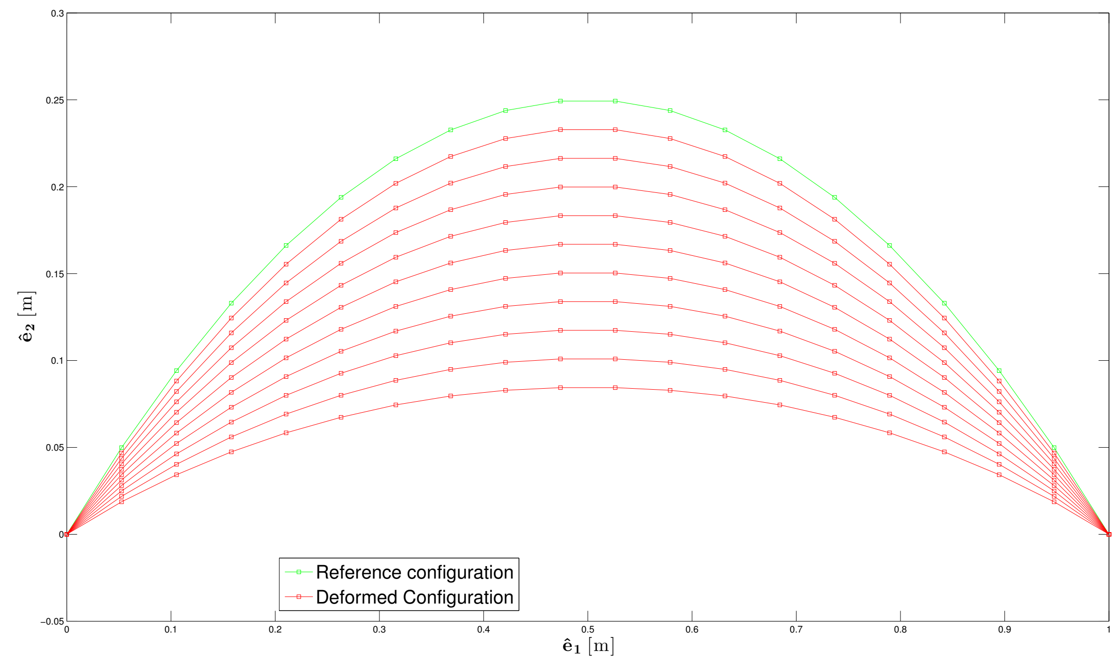

Figure 3: Finite element solution to a 10-element parabolic arch of unit span and $EI = 1$ with a distributed load $q = -2.50$ applied to the center element in 10 steps. Both ends of the arch are simply supported. The deformations are scaled by a factor of 100 in order to visualize the result.

Introducing a test function η as we did before, and integrating gives

$$\int_L EI\frac{d^4u}{dx^4}\eta dx - \int_L \frac{3}{2}EA\left(\frac{du}{dx}\right)^2\frac{d^2u}{dx^2}\eta dx - \int_L q(x)\eta dx = 0$$

Integrating by parts leads to the weak form of the equilibrium equation

$$EI\int_L\left(\frac{d^3u}{dx^3}\eta\right)_{,x}dx - EI\int_L\left(\frac{d^2u}{dx^2}\eta_{,x}\right)_{,x}dx + EI\int_L\frac{d^2u}{dx^2}\eta_{,xx}dx - \frac{3}{2}EA\int_L\frac{d^2u}{dx^2}\left(\frac{du}{dx}\right)^2\eta dx - \int_L q\eta dx = 0$$

$$EI\left[\underbrace{\frac{d^3u}{dx^3}\eta - \frac{d^2u}{dx^2}\eta_{,x}}_{\text{Boundary terms}} + \int_L\frac{d^2u}{dx^2}\eta_{,xx}dx\right] - \frac{3}{2}EA\int_L\frac{d^2u}{dx^2}\left(\frac{du}{dx}\right)^2\eta dx - \int_L q\eta dx = 0$$

Applying the shape functions from before using a Galerkin scheme, we get the FE equations

$$\left[EI\int_L u^a\frac{d^2N^a}{dx^2}\frac{d^2N^b}{dx^2}dx - \frac{3EA}{2}\int_L u^a\frac{d^2N^a}{dx^2}\left(u^c\frac{dN^c}{dx}\right)^2 N^b dx - \int_L qN^b dx - V^*N^b\Big|_L - M^*\frac{dN^b}{dx}\Big|_L\right]\eta^b = 0 \quad \forall\eta^b$$

This needs to be linearized and solved using Newton-Raphson iterations. Taking $u^a + du^a$ to be a

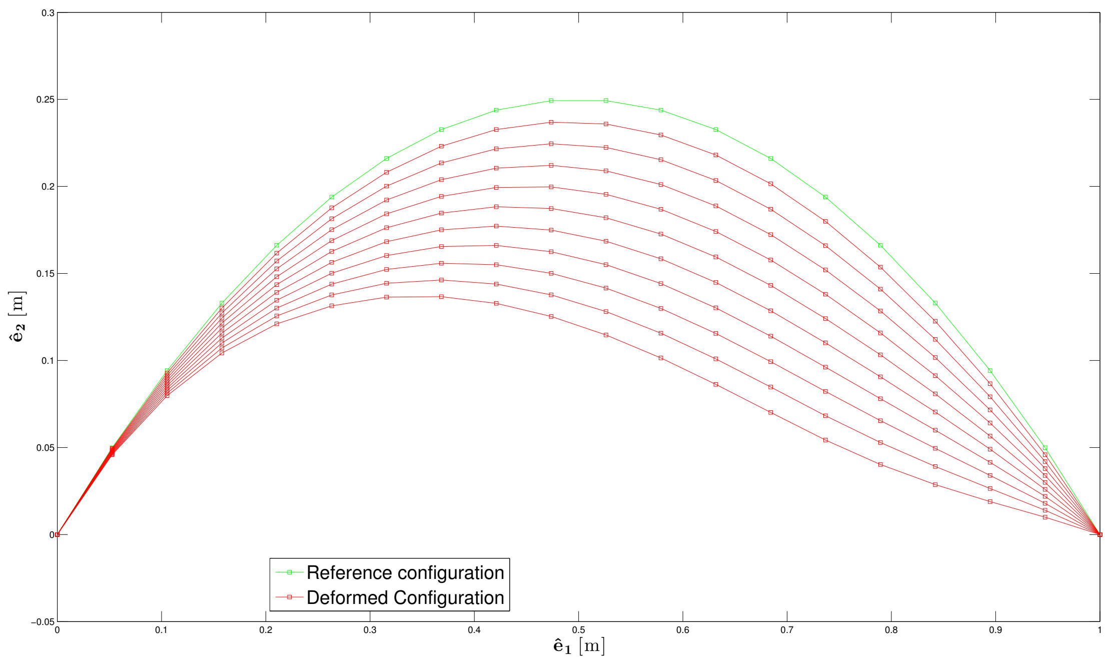

Figure 4: Finite element solution to a 10-element parabolic arch of unit span and $EI = 1$ with a distributed load $q = -2.50$ applied to the element located at $\frac{3}{4}$ span in 10 steps. The left side of the arch is encastered while the right is simply supported. The deformations are scaled by a factor of 100 in order to visualize the result.

solution to the equation, we can expand the nonlinear terms as taylor series keeping only the terms linear in du^a.

$$\int f(u^b + du^b)C^a dx \approx \int \left(f(u^b) + \frac{\partial f}{\partial u^b} du^b \right) C^a dx$$

The FE equation can be expressed as

$$du^b K_{ab} + R_a = Q_a$$

$$\begin{aligned}
R_a &= \int_L \left[EIu^c \frac{d^2N^c}{dx^2}\frac{d^2N^a}{dx^2} - \frac{3EA}{2}\left(u^c\frac{dN^c}{dx}\right)^2 \frac{d^2N^d}{dx^2} u^d N^a \right] dx \\
K_{ab} &= \int_L \left[EI\frac{d^2N^b}{dx^2}\frac{d^2N^a}{dx^2} - \frac{3EA}{2}\frac{d^2N^b}{dx^2} N^a \left(u^c\frac{dN^c}{dx}\right)^2 - 3EA\frac{d^2N^c}{dx^2}u^c\frac{dN^d}{dx}u^d\frac{dN^b}{dx}N^a \right] dx \\
Q_a &= \int_L qN^a dx + V^* N^a\Big|_L + M^* \frac{dN^a}{dx}\Big|_L
\end{aligned}$$

Following implementation in MatLab, I compared the solution of a cantilever beam with a distributed transverse load (Figure **??**) to the solution from the linear code given in Figure 1. The displacements agree with the linear solution for small loads. Although this could be interpreted as a validation of the code, there are convergence problems when applying larger loads that would result in finite scale deformations. This has prevented me from getting solutions for instabilities commonly associated with nonlinear mechanics such as snap-through buckling.

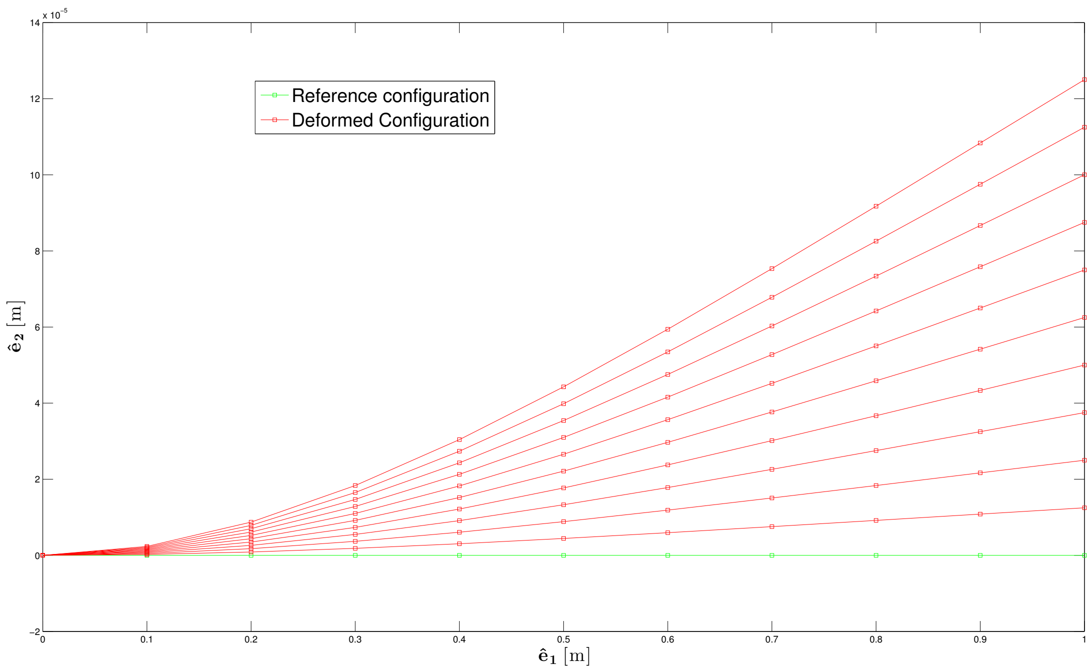

Figure 5: Finite element solution to a 10-element cantilever of unit length and $EI = 1$ using a nonlinear beam formulation. A distributed load $q = 0.001$ is applied over 10 steps.