EN234: Three-dimentional Timoshenko beam element undergoing axial, torsional and bending deformations

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1 Introduction

Timoshenko beam theory is applied to discribe the behaviour of short beams when the cross-sectional dimensions of the beam are not small compared to its length. Here in this project, we develop the theoretical formulation for three-dimentional Timoshenko beam element undergoing axial, torsional and bending deformations. Then we implement it into EN234FEA.

2 The kinemation relations

Consider a typical two-node beam element of length l, where each node has six degrees of freedom. In local reference system, the beam is along with x-axis. The elastic deformation vector of the beam element is $\mathbf{d} = [U\ V\ W]^T$ that can be expressed as:

$$\begin{aligned} U &= u - y\frac{\partial v}{\partial x} - z\frac{\partial w}{\partial x}, \\ V &= -z\theta_x + v, \\ W &= y\theta_x + w. \end{aligned} \tag{1}$$

where u is the axial stretch displacment, v, w consist of contributions v_b, w_b and v_s, w_s due to bending and transverse shear, that is

$$v = v_b + v_s\ ,\ w = w_b + w_s. \tag{2}$$

The relationships between total slope, bending rotation and transverse shear are

$$\begin{aligned} \frac{\partial v}{\partial x} &= \frac{\partial v_b}{\partial x} + \frac{\partial v_s}{\partial x} = \theta_z + \gamma_{xy}, \\ \frac{\partial w}{\partial x} &= \frac{\partial w_b}{\partial x} + \frac{\partial w_s}{\partial x} = -\theta_y + \gamma_{xz}, \end{aligned} \tag{3}$$

where γ_{xy} and γ_{xz} are shear strains in the (xy)- and (xz)-planes, respectively. The two rotations θ_y, θ_z are related to the bending deformations v_b, w_b by the expressions

$$\theta_z = \frac{\partial v_b}{\partial x}\ ,\ \theta_y = -\frac{\partial w_b}{\partial x} \tag{4}$$

The strain components are

$$\varepsilon_{xx} = u_x - y\frac{d\theta_z}{dx} + z\frac{d\theta_y}{dx} \ , \ \gamma_{xy} = \frac{dv}{dx} - \theta_z \ , \ \gamma_{xz} = \frac{dw}{dx} + \theta_y \tag{5}$$

3 Derivation of shape functions

Shape function matrices for axial and torsional deformation are given by[1]

$$[N_u] = [N_{\theta_x}] = [(1-\xi) \ \ \xi], \tag{6}$$

where $\xi = x/l$ is the dimensionless axial coordinate. Shape functions for bending deformation in the (xy)-plane are derived as follows:

$$v(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \tag{7}$$

The shear strain is assumed to be constant along the finite element $\gamma_{xy} = \gamma_0$.[2] Thus the slope due to bending can be obtained:

$$\theta_z = a_1 + 2a_2 x + 3a_3 x^2 - \gamma_0 \tag{8}$$

and the moment-curvature relationship is

$$M_z = -EI_{zz}\frac{\partial\theta_z}{\partial x} = -EI_{zz}(2a_2 + 6a_3 x) \tag{9}$$

the shear force is related to the transverse shear strain by

$$Q_y = \kappa G A \gamma_{xy} \tag{10}$$

where κ is the shear correction factor that accounts for the non-uniform distribution of the shear stress over the cross-section A; E is the modulus of elasticity, and G is the shear modulus, I_{zz} is the second moment of area about the z-axis. The bending moment M_z and the shearing force Q_y are related by

$$\frac{dM_z}{dx} - Q_y = 0 \tag{11}$$

Combining which equation (9) and (10), we get the expression of γ_0

$$\gamma_0 = -6\frac{EI_{zz}}{\kappa GA}a_3 = -6\Lambda_z a_3 \ , \ \Lambda_z = \frac{EI_{zz}}{\kappa GA} \tag{12}$$

The following boundary conditions must be satisfied:

$$v(0) = v_1 \ , \ v(l) = v_2 \ , \ \theta(0) = \theta_{z1} \ , \ \theta(l) = \theta_{z2} \tag{13}$$

This can be written in matrix form

$$\begin{Bmatrix} v_1 \\ \theta_{z1} \\ v_2 \\ \theta_{z2} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6\Lambda_z \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & (3l^2 + 6\Lambda_z) \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} \tag{14}$$

Or in more compact form

$$\mathbf{d} = \mathbf{A}\mathbf{a} \tag{15}$$

Sloving for **a** and substituting the values of a_j into the expression of $v(\xi = x/l)$ and simplifying, we obtains

$$v(\xi) = N_{v_1} v_1 + N_{v_2} \theta_{z1} + N_{v_3} v_2 + N_{v_4} \theta_{z2} \tag{16}$$

where

$$\begin{aligned}
N_{v_1} &= \overline{\Phi}_z \left(1 - 3\xi^2 + 2\xi^3 + \Phi_z(1-\xi)\right), \\
N_{v_2} &= l\overline{\Phi}_z \left(\xi - 2\xi^2 + \xi^3 + \Phi_z(\xi - \xi^2)/2\right), \\
N_{v_3} &= \overline{\Phi}_z \left(3\xi^2 - 2\xi^3 + \Phi_z \xi\right), \\
N_{v_4} &= l\overline{\Phi}_z \left(-\xi^2 + \xi^3 + \Phi_z(-\xi + \xi^2)/2\right).
\end{aligned} \tag{17}$$

Hence $\theta_z(\xi)$ can be written in the form

$$\theta_z(\xi) = N_{\theta_{z1}} v_1 + N_{\theta_{z2}} \theta_{z1} + N_{\theta_{z3}} v_2 + N_{\theta_{z4}} \theta_{z2} \tag{18}$$

where

$$\begin{aligned}
N_{\theta_{z1}} &= 6\frac{\overline{\Phi}_z}{l} \left(-\xi + \xi^2\right), \\
N_{\theta_{z2}} &= \overline{\Phi}_z \left(1 - 4\xi + 3\xi^2 + \Phi_z(1-\xi)\right), \\
N_{\theta_{z3}} &= -6\frac{\overline{\Phi}_z}{l} \left(-\xi + \xi^2\right), \\
N_{\theta_{z4}} &= \overline{\Phi}_z \left(-2\xi + 3\xi^2 + \Phi_z \xi\right).
\end{aligned} \tag{19}$$

where

$$\overline{\Phi}_z = \frac{1}{1 + \Phi_z}, \; \Phi_z = \frac{12\Lambda_z}{l^2} = \frac{12EI_{zz}}{\kappa G A l^2} \tag{20}$$

Shape functions for bending in the (xz)-plane are obtained in a similar manner

$$\overline{\Phi}_y = \frac{1}{1 + \Phi_y}, \; \Phi_y = \frac{12EI_{yy}}{\kappa G A l^2} \tag{21}$$

$$\begin{aligned}
N_{w_1} &= \overline{\Phi}_y \left(1 - 3\xi^2 + 2\xi^3 + \Phi_y(1-\xi)\right), \\
N_{w_2} &= -l\overline{\Phi}_y \left(\xi - 2\xi^2 + \xi^3 + \Phi_y(\xi - \xi^2)/2\right), \\
N_{w_3} &= \overline{\Phi}_y \left(3\xi^2 - 2\xi^3 + \Phi_y \xi\right), \\
N_{w_4} &= -l\overline{\Phi}_y \left(-\xi^2 + \xi^3 + \Phi_y(-\xi + \xi^2)/2\right).
\end{aligned} \tag{22}$$

$$\begin{aligned}
N_{\theta_{y1}} &= 6\frac{\overline{\Phi}_y}{l} \left(-\xi + \xi^2\right), \\
N_{\theta_{y2}} &= -\overline{\Phi}_y \left(1 - 4\xi + 3\xi^2 + \Phi_y(1-\xi)\right), \\
N_{\theta_{y3}} &= -6\frac{\overline{\Phi}_y}{l} \left(-\xi + \xi^2\right), \\
N_{\theta_{y4}} &= -\overline{\Phi}_y \left(-2\xi + 3\xi^2 + \Phi_y \xi\right).
\end{aligned} \tag{23}$$

By the above shape functions, the kinematic relation can be expressed as

$$\{\mathbf{d}\}_{6\times1} = [U \quad V \quad W \quad \Theta_x \quad \Theta_y \quad \Theta_z]^T = [\mathbf{N}]_{6\times12}\{\mathbf{e}\}_{12\times1} \tag{24}$$

4 Element stiffness and residual force

The total potential energy

$$\begin{aligned}\Pi_P =& \int_0^l \frac{1}{2}EI_{yy}(\frac{d\theta_z}{dx})^2dx + \int_0^l \frac{1}{2}EI_{zz}(\frac{d\theta_y}{dx})^2dx + \int_0^l \frac{1}{2}GA\kappa(\gamma_{xy}^2+\gamma_{xz}^2)dx \\ &+ \int_0^l \frac{1}{2}EA\kappa(\frac{du}{dx})^2dx + \int_0^l \frac{1}{2}GJ_\rho(\frac{d\theta_x}{dx})^2dx - \int_0^l q_v v dx - \int_0^l q_w w dx \\ &- Q_v^* v\mid_l - Q_w^* w\mid_l - M_y^*\theta_y\mid_l - M_z^*\theta_z\mid_l - T_x^*\theta_x\mid_l - N_x^* u\mid_l\end{aligned} \tag{25}$$

Do finite element discretization. We get the local element stiffness matrix and the residual force vetcor formed as

$$\mathbf{K} = \int_0^1 \left(EI_{yy}\mathbf{B_{by}}^T\mathbf{B_{by}} + EI_{zz}\mathbf{B_{bz}}^T\mathbf{B_{bz}} + GA\kappa(\mathbf{B_{sy}}^T\mathbf{B_{sy}} + \mathbf{B_{sz}}^T\mathbf{B_{sz}})\right)d\xi + \int_0^1 \left(EA\mathbf{B_s}^T\mathbf{B_s} + GJ_\rho\mathbf{B_r}^T\mathbf{B_r}\right)d\xi \tag{26}$$

$$\mathbf{R} = \int_0^1 q\mathbf{N}^T d\xi \quad , \quad \mathbf{F} = \mathbf{N^T f}\mid_0^1 \tag{27}$$

The degrees of freedom are found by solving the linear equation

$$\mathbf{Ku} = \mathbf{R} + \mathbf{F} \tag{28}$$

Where $\mathbf{K}$ can be sloved out analytically[3]

$$\begin{bmatrix}
\frac{EA}{l} & & & & & & & & & & & \\
0 & \frac{12\overline{\Phi}_z EI_{zz}}{l^3} & & & & & & & & & & \\
0 & 0 & \frac{12\overline{\Phi}_y EI_{yy}}{l^3} & & & & & & & & & \\
0 & 0 & 0 & \frac{GJ_\rho}{l} & & & & & symmetry & & & \\
0 & 0 & -\frac{6\overline{\Phi}_y EI_{yy}}{l^2} & 0 & \frac{(4+\Phi_y)\overline{\Phi}_y EI_{yy}}{l} & & & & & & & \\
0 & \frac{6\overline{\Phi}_z EI_{zz}}{l^2} & 0 & 0 & 0 & \frac{(4+\Phi_z)\overline{\Phi}_z EI_{zz}}{l} & & & & & & \\
-\frac{EA}{l} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{l} & & & & & \\
0 & -\frac{12\overline{\Phi}_z EI_{zz}}{l^3} & 0 & 0 & 0 & -\frac{6\overline{\Phi}_z EI_{zz}}{l^2} & 0 & \frac{12\overline{\Phi}_z EI_{zz}}{l^3} & & & & \\
0 & 0 & -\frac{12\overline{\Phi}_y EI_{yy}}{l^3} & 0 & \frac{6\overline{\Phi}_y EI_{yy}}{l^2} & 0 & 0 & 0 & \frac{12\overline{\Phi}_y EI_{yy}}{l^3} & & & \\
0 & 0 & 0 & -\frac{GJ_\rho}{l} & 0 & 0 & 0 & 0 & 0 & \frac{GJ_\rho}{l} & & \\
0 & 0 & -\frac{6\overline{\Phi}_y EI_{yy}}{l^2} & 0 & \frac{(2-\Phi_y)\overline{\Phi}_y EI_{yy}}{l} & 0 & 0 & 0 & \frac{6\overline{\Phi}_y EI_{yy}}{l^2} & 0 & \frac{(4+\Phi_y)\overline{\Phi}_y EI_{yy}}{l} & \\
0 & \frac{6\overline{\Phi}_z EI_{zz}}{l^2} & 0 & 0 & 0 & \frac{(2-\Phi_z)\overline{\Phi}_z EI_{zz}}{l} & 0 & -\frac{6\overline{\Phi}_z EI_{zz}}{l^2} & 0 & 0 & 0 & \frac{(4+\Phi_z)\overline{\Phi}_z EI_{zz}}{l}
\end{bmatrix} \tag{29}$$

For uniformly distributed load q_{v0}, q_{w0}, the residual vector yields

$$\mathbf{R} = \frac{l}{12}[0 \quad 6q_{v0} \quad 6q_{w0} \quad 0 \quad -lq_{w0} \quad lq_{v0} \quad 0 \quad 6q_{v0} \quad 6q_{w0} \quad 0 \quad lq_{w0} \quad -lq_{v0}]^T \tag{30}$$

5 The transformation between local reference system and global reference system

The above formulation is only valid in local reference system. For every element, we need to do coordinate transformation from global reference system to local reference system. The degree of freedom of one element in global reference system is

$$\mathbf{u} = [u_1 \quad v_1 \quad w_1 \quad \theta_{x1} \quad \theta_{y1} \quad \theta_{z1} \quad u_2 \quad v_2 \quad w_2 \quad \theta_{x2} \quad \theta_{y2} \quad \theta_{z2}] \tag{31}$$

The degree of freedom of one element in local reference system is

$$\mathbf{u}' = [u'_1 \quad v'_1 \quad w'_1 \quad \theta'_{x1} \quad \theta'_{y1} \quad \theta'_{z1} \quad u'_2 \quad v'_2 \quad w'_2 \quad \theta'_{x2} \quad \theta'_{y2} \quad \theta'_{z2}] \tag{32}$$

The tranformation matirx is

$$\mathbf{Q} = \begin{pmatrix} T & & & \\ & T & & \\ & & T & \\ & & & T \end{pmatrix}_{12\times 12} , \quad \mathbf{T} = \begin{bmatrix} cos(x'x) & cos(x'y) & cos(x'z) \\ cos(y'x) & cos(y'y) & cos(y'z) \\ cos(z'x) & cos(z'y) & cos(z'z) \end{bmatrix} \tag{33}$$

where T satisfies

$$\begin{Bmatrix} u'_i \\ v'_i \\ w'_i \end{Bmatrix} = [\mathbf{T}] \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} , \begin{Bmatrix} \theta'_{xi} \\ \theta'_{yi} \\ \theta'_{zi} \end{Bmatrix} = [\mathbf{T}] \begin{Bmatrix} \theta_{xi} \\ \theta_{yi} \\ \theta_{zi} \end{Bmatrix} , \ i = 1,2 \tag{34}$$

Thus, the transformation from global reference system to local reference system of displacment, force and element stiffness matrix can be expressed as

$$\mathbf{u}' = \mathbf{Q}\mathbf{u} , \ \mathbf{F}' = \mathbf{Q}\mathbf{F} , \ \mathbf{K}' = \mathbf{Q}\mathbf{K}\mathbf{Q}^{\mathbf{T}} \tag{35}$$

The method to determine $\mathbf{T}$ is discussed as below. Firstly, we need to slove the components of axial vector $\mathbf{i}$ in the global reference system. Given the global coordinates of the beam $(x_1, y_1, z_1, x_2, y_2, z_2)$, the first row of $\mathbf{T}$ is

$$T_{11} = \frac{x_2 - x_1}{l} , \ T_{12} = \frac{y_2 - y_1}{l} , \ T_{13} = \frac{z_2 - z_1}{l} \tag{36}$$

where

$$l = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{37}$$

Then given the cross-section direction vector in global reference system $\mathbf{k}' = (k'_1, k'_2, k'_3)$ (This is define in the input file along with the second moment of area I_{yy}, I_{zz}), the local (xz)-plane is formed by vectors $\mathbf{i}$ and $\mathbf{k}'$. we can slove vector $\mathbf{j} = -\mathbf{i} \times \mathbf{k}'$, thus the second row of $\mathbf{T}$ is

$$T_{21} = -\frac{T_{12}k'_3 - T_{13}k'_2}{A} , \ T_{22} = -\frac{T_{13}k'_1 - T_{11}k'_3}{A} , \ T_{23} = -\frac{T_{11}k'_2 - T_{12}k'_1}{A} \tag{38}$$

where

$$A = \sqrt{(T_{12}k'_3 - T_{13}k'_2)^2 + (T_{13}k'_1 - T_{11}k'_3)^2 + (T_{11}k'_2 - T_{12}k'_1)^2} \tag{39}$$

Now the normal vector $\mathbf{k}$ of local (xy)-plane is given by $\mathbf{k} = \mathbf{i} \times \mathbf{j}$. Therefore the third row of $\mathbf{T}$ is given by

$$T_{31} = \frac{T_{12}T_{23} - T_{13}T_{22}}{B} , \ T_{32} = -\frac{T_{13}T_{21} - T_{11}T_{23}}{B} , \ T_{33} = -\frac{T_{11}T_{22} - T_{12}T_{21}}{B} \tag{40}$$

where

$$B = \sqrt{(T_{12}T_{23} - T_{13}T_{22})^2 + (T_{13}T_{21} - T_{11}T_{23})^2 + (T_{11}T_{22} - T_{12}T_{21})^2} \quad (41)$$

6 EN234FEA implement

The twelve properties we difined for one beam element is

- E, G Young's modulus and shear modulus
- A The area of cross section
- k' The position vector of local z-direction
- I_{zz}, I_{yy}, J_ρ The second moments of area about the z-axis and the y-axis, polar moment of area
- κ Shear correction factor depending on the shape of the cross section, typically 5/6 for rectangular section
- l_y, h_z The maxima distance from the edge to the center of the cross section in y-direction and z-direction
- q_{v0}, q_{w0} Uniformly distributed load on y-direction and z-direction

In our code, we plan to calculate the following state variables and output them by user_print subroutine:

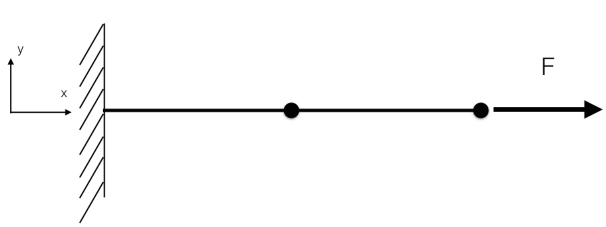
- S_1 Normal stress due to stretch
- T_1 Maximum shear stress due to torsion
- S_2 Maximum Normal stress due to bending in y-direction
- T_2 Shear stress due to bending in y-direction
- S_3 Maximum Normal stress due to bending in z-direction
- T_3 Shear stress due to bending in z-direction

They can be calculated from the following formulas,

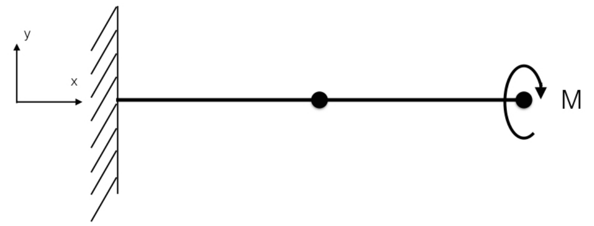
$$\begin{aligned}
S_1 &= \frac{E(u_2 - u_1)}{l} \\
T_1 &= \frac{G(\theta_{x2} - \theta_{x1})}{l} max(l_y, h_z) \\
S_2 &= \frac{El_y(\theta_{z2} - \theta_{z1})}{l} \\
T_2 &= G\gamma_{0y} = \frac{-G\Phi_z\overline{\Phi}_z(2v_1 + \theta_{z1}l - 2v_2 + \theta_{z2}l)}{2l} \\
S_3 &= \frac{Eh_z(\theta_{y2} - \theta_{y1})}{l} \\
T_3 &= G\gamma_{0z} = \frac{-G\Phi_y\overline{\Phi}_y(2w_1 + \theta_{y1}l - 2w_2 + \theta_{y2}l)}{2l}
\end{aligned} \quad (42)$$

7 The test problems

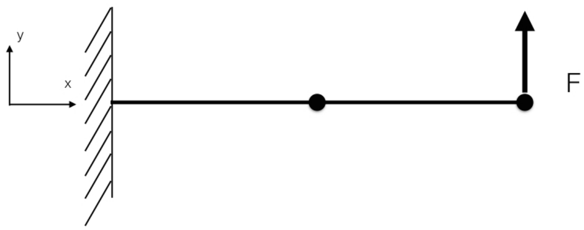
To verify the code, I'll do several tests with axial, torsional and bending deformations. The bending loads contain distributed force, focus shear force and force moment.



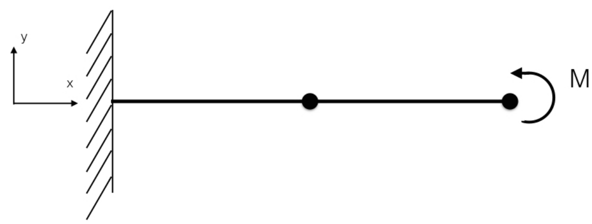
(a) Axial stretch test



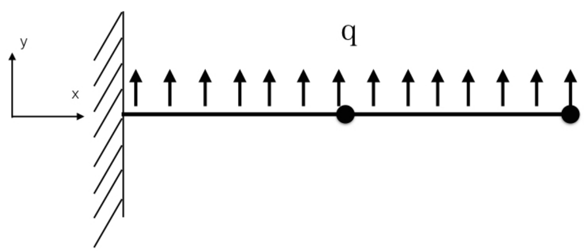
(b) Tortion test



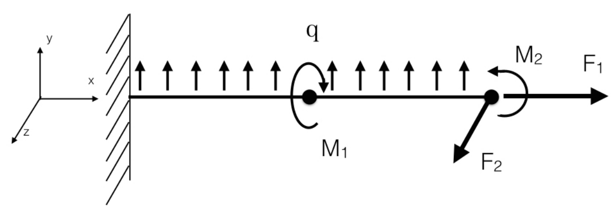
(c) bending test with point load



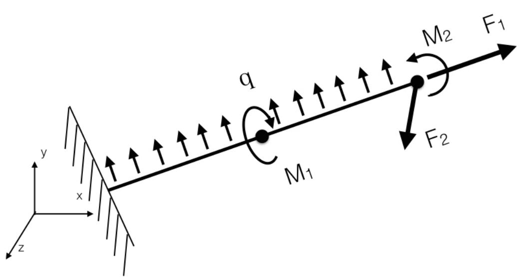
(d) bending test with moment



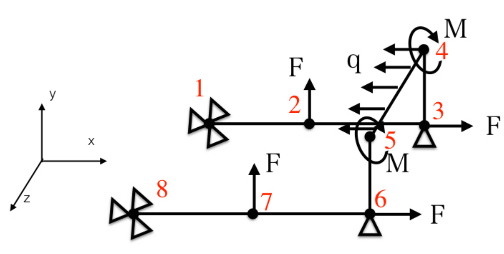
(e) bending test with distributed load



(f) Test with all the above load



(g) Test with all the above load in 3d space



(h) Complex Test in 3d space

Figure 1: Test models

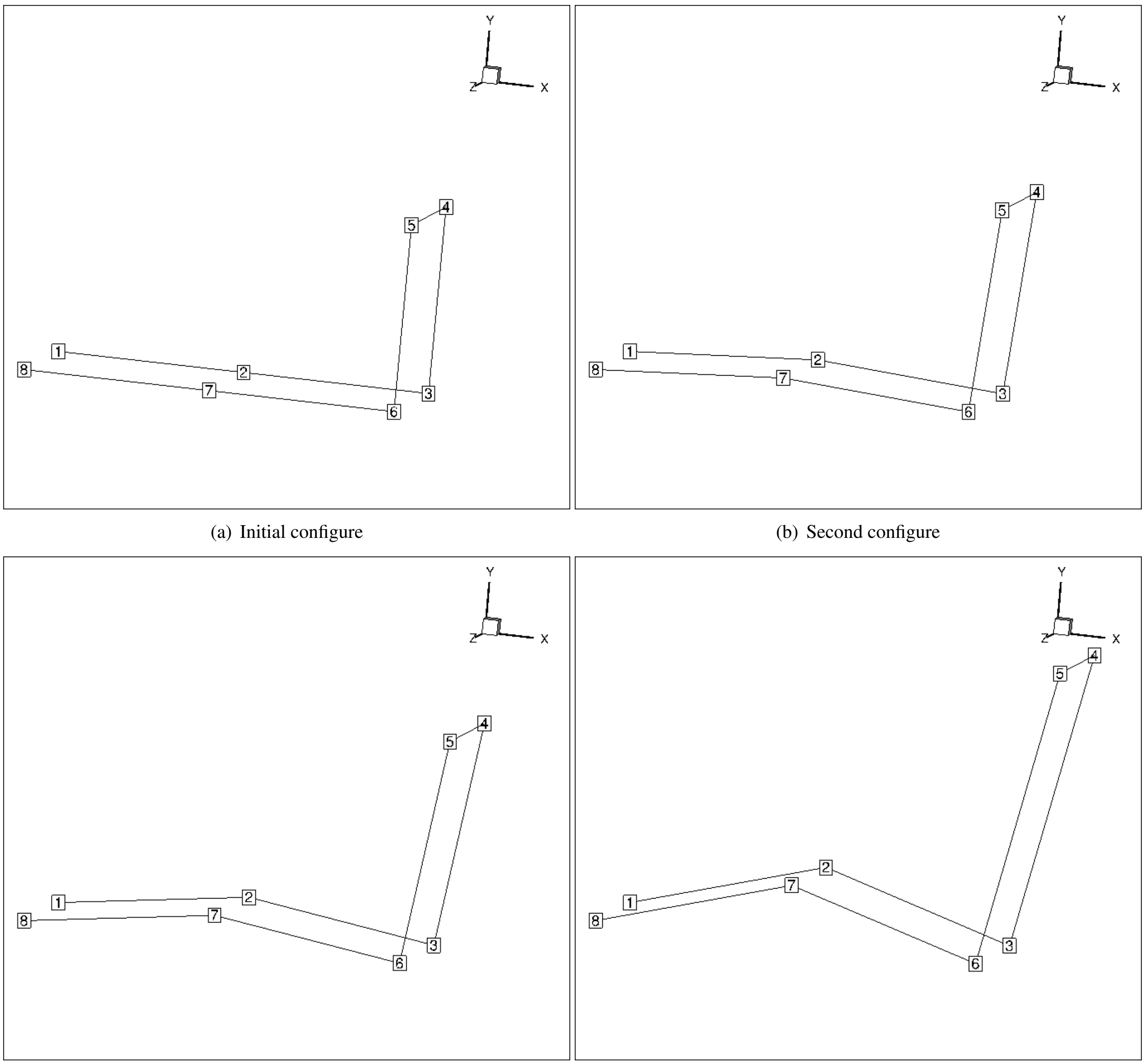
In the first step, different kinds of forces are applied seperately. The analytical solution can be derived easily and the numerial solutions agree with them very well.

In the second step, these loads are applied on the elements at the same time. The deformation and stresses also superpose linearly since it is under small deformation.

In the third step, I make a rotation to let the beam not along with x-axis. By appling the same loads, I verify that the stresses in local coordinate system of the beam do not change.

In the fourth step, I build a complex model without analytical solution. By plotting the deformation configures in Tecplot, I infer that the result is reasonable qualitatively.

The input files and result data files of all the tests are available in the beam file directory.



(a) Initial configure

(b) Second configure

(c) Third configure

(d) Final configure

Figure 2: Tecplot visualization of the solution of the last test

References

[1] A Bazoune, YA Khulief, and NG Stephen. Shape functions of three-dimensional timoshenko beam element. *Journal of Sound and Vibration*, 259(2):473–480, 2003.

[2] R Davis, RD Henshell, and GB Warburton. A timoshenko beam element. *Journal of Sound and Vibration*, 22(4):475–487, 1972.

[3] Janusz S Przemieniecki. *Theory of matrix structural analysis*. Courier Corporation, 1985.