Hyperelastic Material Collision – Galilean Cannon

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1 Project introduction

This project will study an interesting project – the Galilean Cannon. This is a multiple ball system where several balls are stacked aligning their centers along the same axis as shown in Fig. 1. These balls, heavier ones in lower positions, will drop down to the ground under gravity enforcement after they are set free. The surprising phenomenon will happen when these balls rebound: the top ball will go up to the height that is much higher than its original height. Basically the more balls we use, the higher the top ball can reach. Therefore, this system can launch a ball like a cannon, which may indicate the source of its name.

This phenomenon can be explained by various mechanical and physical methods and models [1-3]. For example, the balls can be regarded as perfect elastic bodies who will not lose any energy after collision and whose deformation is neglected. In this way, the motions of balls can be predicted by principles of energy and linear momentum conservations. Alternatively, linear elastic and finite deformation material models can also be used to analyze this problem, which will be more complex.

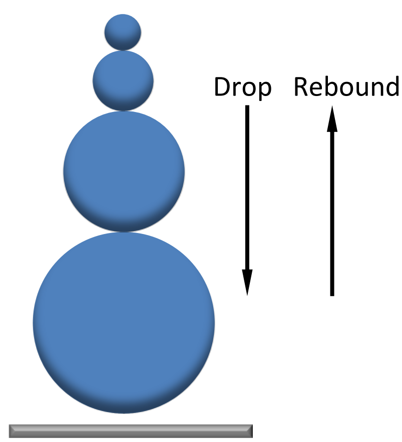


Fig. 1 Schematic illustration of the Galilean Cannon

This project is going to study and simulate the hyperelastic behavior of this stacking balls system both theoretically and numerically, predict the final height the top ball can reach under different conditions and optimize the optimal mass distribution of balls to make the height as large as possible. The rest of the project will be organized as follows. In Section 2, perfect elastic collision theory will be used to analyze this system, and prediction on the final height considering energy loss during collision can be given. In Section 3, finite element analysis by using hyperelastic material in ABAQUS is implemented for different cases. Optimization problems will be investigated in Section 4 for both rigid model and finite element model. Finally, some concluding remarks will be made in the last section.

2 Elastic analysis and prediction

Since what we care about most is the final height of the top ball which is only determined by the velocity of the ball after collisions, the intermediate movement and deformation processes can be modeled in different ways. In this section, we first study this stacking ball system by theoretical analysis which can approximately give us an understanding to this problem and help us find out some intrinsic mechanisms from the phenomenon by simplifying complex collisions.

Before using any theoretical tools to resolve this problem, we should make some assumptions for our physical/mechanical model in order to simplify the problem so that analytical solutions are possible to obtain. The assumptions include:

1) All the balls are dropped from a certain height with initial velocities 0;
2) Gravity is the only force imposed on the balls except during collision processes;
3) During collisions, the gravity is neglected comparing with contact forces between balls;
4) Diameters of these balls can be neglected compared with the height of them from the ground, namely they are in the same height when dropped;
5) At the beginning of the moment of any collision, velocities of all balls are the same;
6) The energy dissipation ratio of every collision (includes collision between balls as well as the collision between balls and the ground) is an invariant;
7) The collision between the largest ball and the ground happens first, then the collision between the largest ball and the second largest ball, and so on;
8) After each collision between two balls, the velocity of the smaller ball will change its direction no matter whether the velocity of the other one will or not.

Next let us consider two cases: one is the collision between the heaviest ball and the ground, and the other one is the collision between two adjacent balls. Suppose the energy dissipation ratio of every collision is α. The two adjacent balls included in the collision are labelled as p and q, so their masses are m_p and m_q ($m_p > m_q$), respectively. Velocities of the two balls before collision are v_{p0} and v_{q0}, respectively, and after collision v_{p1} and v_{q1}, respectively.

We can do some scaling treatment for variables in the model. Suppose the mass, velocity of the largest ball before collision with the ground and velocity after collision with the ground are m_a, v_{a0} and v_{a1}, respectively. Then let

$$\widehat{m}_p = \frac{m_p}{m_a}, \widehat{m}_q = \frac{m_q}{m_a}, \tag{2.1a}$$

$$\hat{v}_{p0} = \frac{v_{p0}}{v_{a0}}, \hat{v}_{q0} = \frac{v_{q0}}{v_{a0}}, \hat{v}_{p1} = \frac{v_{p1}}{v_{a1}}, \hat{v}_{q1} = \frac{v_{q1}}{v_{a1}}. \tag{2.1b}$$

I. For the first case, namely the collision between the heaviest ball and the ground, we have

$$\frac{1}{2}\hat{m}_p \hat{v}_{p0}^2 \cdot \alpha = \frac{1}{2}\hat{m}_p \hat{v}_{p1}^2, \tag{2.2}$$

thus,

$$\hat{v}_{p1} = -\sqrt{\alpha}\hat{v}_{p0}. \tag{2.3}$$

II. For the second case, namely the collision between two balls, according to the conservation of linear momentum and conservation of kinematic energy, we have

$$\hat{m}_p \hat{v}_{p0} + \hat{m}_q \hat{v}_{q0} = \hat{m}_p \hat{v}_{p1} + \hat{m}_q \hat{v}_{q1}, \tag{2.4}$$

$$\left(\frac{1}{2}\hat{m}_p \hat{v}_{p0}^2 + \frac{1}{2}\hat{m}_q \hat{v}_{q0}^2\right) \cdot \alpha = \frac{1}{2}\hat{m}_p \hat{v}_{p1}^2 + \frac{1}{2}\hat{m}_q \hat{v}_{q1}^2. \tag{2.5}$$

From Eqs. (2.4) and (2.5) we can get

$$\begin{aligned}(\hat{m}_p + \hat{m}_q)\hat{m}_q \hat{v}_{q1}^2 - 2\hat{m}_q(\hat{m}_p \hat{v}_{p0} + \hat{m}_q \hat{v}_{q0})\hat{v}_{q1} + (1-\alpha)\hat{m}_p^2 \hat{v}_{p0}^2 \\ +(\hat{m}_q - \alpha \hat{m}_p)\hat{m}_q \hat{v}_{q0}^2 + 2\hat{m}_p \hat{m}_q \hat{v}_{p0}\hat{v}_{q0} = 0.\end{aligned} \tag{2.6}$$

Eq. (2.6) is a quadratic equation with respect to $\hat{v}_{q1}$ which is the rebounded velocity of the ball q which is the smaller ball. The solution of Eq. (2.6) can be written in the following form:

$$\hat{v}_{q1} = \frac{(\hat{m}_p \hat{v}_{p0} + \hat{m}_q \hat{v}_{q0}) \pm \sqrt{(\hat{m}_p \hat{v}_{p0} + \hat{m}_q \hat{v}_{q0})^2 - (\hat{m}_p + \hat{m}_q)\beta}}{\hat{m}_p + \hat{m}_q}, \tag{2.7a}$$

where

$$\beta = \frac{1}{\hat{m}_q}\left[(1-\alpha)\hat{m}_p^2 \hat{v}_{p0}^2 + (\hat{m}_q - \alpha \hat{m}_p)\hat{m}_q \hat{v}_{q0}^2 + 2\hat{m}_p \hat{m}_q \hat{v}_{p0}\hat{v}_{q0}\right]. \tag{2.7b}$$

In solution (2.7), only the one that satisfies the Assumption (8) is the true velocity of the ball q.

Now suppose the initial height of these stacking balls from the ground is H_0, so the final scaling height that the ball q can reach after collision is

$$\hat{H}_q = \frac{H_q}{H_0} = \hat{v}_{q1}^2. \tag{2.8}$$

If we have a three-ball system (balls are marked as A, B and C) and choose $\alpha = 0.8$, $\widehat{m}_A = 1.0$, $\widehat{m}_B = 0.25$, $\widehat{m}_C = 0.0625$, $\hat{v}_{A0} = \hat{v}_{B0} = \hat{v}_{C0} = -1.0$ and $\hat{v}_{B0} = 1.0$, from Eq. (2.3), (2.7) and (2.8) we can get $\widehat{H}_d = \hat{v}_{C1}^2 = 8.39$ which is much higher than the original height.

3 Finite element simulations

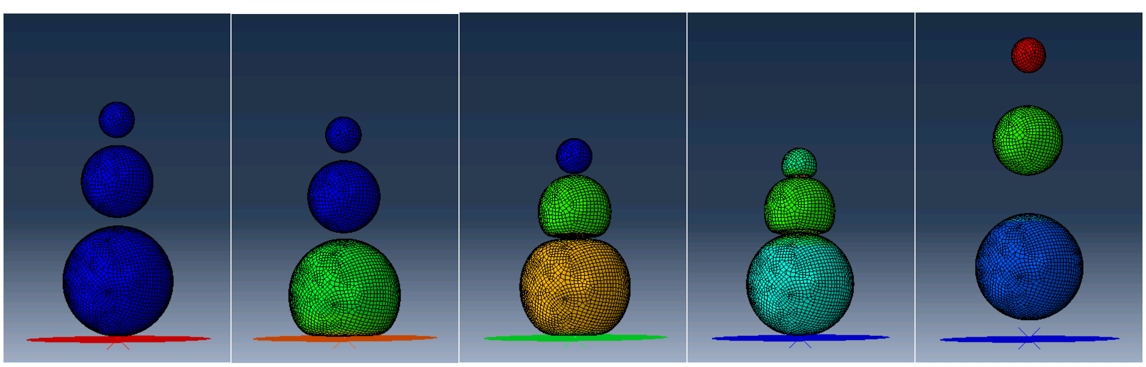
Although we introduce the energy dissipation factor α in Eq. (2.7) to approach real model, the energy loss in each collision for balls with different sizes and masses does not keep such an easy relation with the total kinematic energy. To model the stacking ball system more accurately, in this section we will use finite element analysis to solve this problem in ABAQUS.

We assume the balls to be made of hyperelastic material (i.e., rubber) which is able to bear large elastic deformation. In addition, the balls are modeled as spherical shells but not solid spheres in order to save computational costs. As the seventh assumption in Section 2, we consider the collisions happen in sequence but not at the same time, which is helpful for us to make clear the propagation of stress wave and transmission of linear momentum.

For the parameters of material, we choose those similar to parameters of rubbers. The density of the material is $\rho = 1000$. The constitutive law is chosen as Neo-Hookean relation:

$$w = C_{10}(I_1 - 3) = \frac{\mu}{2}(I_1 - 3), \tag{3.1}$$

thus we have $C_{10} = \mu/2 = E/(1+\nu)/4$, where E and ν are Young's modulus and Poisson's ratio of the material, respectively. Noting that rubber is considered as incompressible material in most cases, so $\nu \approx 0.5$ and $C_{10} \approx E/6$. According to Young's modulus of rubber, we usually choose $10^5 \leq C_{10} \leq 10^{10}$.

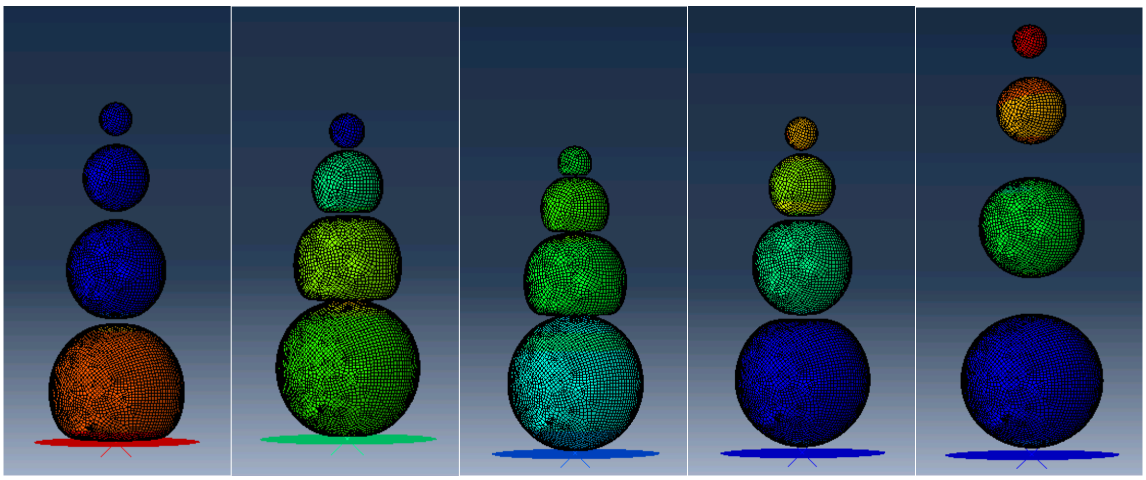


(a) (b) (c) (d) (e)

Fig. 2 The collision process of a three-ball case

Fig. 2a-2e shows a three-ball case where the radii of balls are 3.0, 2.0 and 1.0, and thicknesses 0.3, 0.2 and 0.1, respectively. The coefficient $C_{10} = 1.0 \times 10^5$, and initial velocities of all balls (velocity at the moment before collisions with the ground) are $v_0 = -1.0$. From these figures we can find this hyperelastic model can basically simulate the collision process. After all collisions the top ball will obtain the largest launch velocity.

Similarly we can also solve the problems including more stacking balls, for example Fig. 3a-3e shows the case where four balls with radii $4.0, 3.0, 2.0$ and 1.0 are considered. Their thicknesses are $0.4, 0.3, 0.2$ and 0.1, respectively. The coefficient in the constitutive law is $C_{10} = 1 \times 10^7$, and the initial velocity is $v_0 = -1.0$.



(a) (b) (c) (d) (e)

Fig. 3 The collision process of a four-ball case

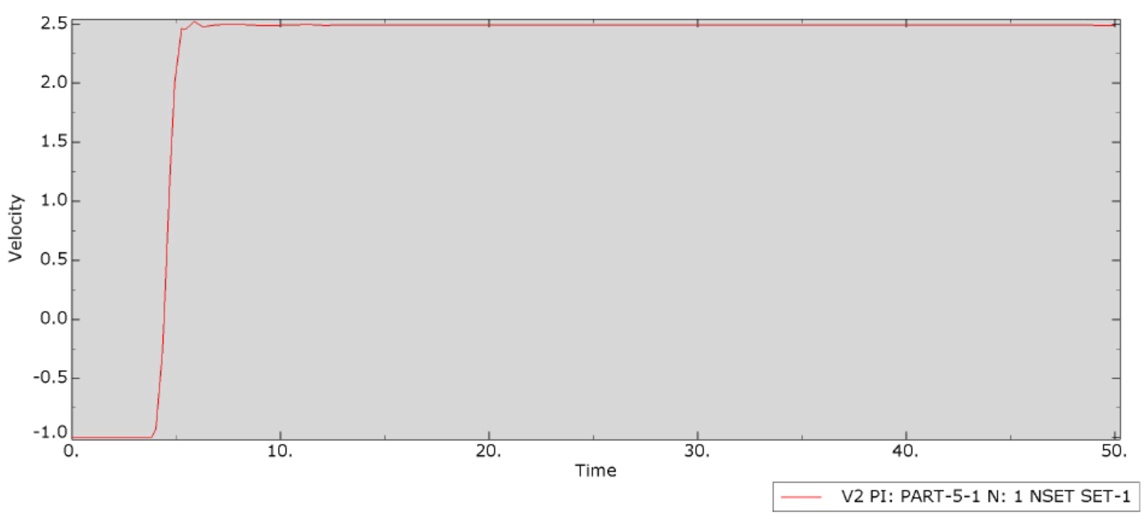


Fig. 4 The velocity history of the top ball in the four-ball case

Next we compare results from the analytical mode in Section 2 ($\alpha = 0.8$) and results from numerical simulation in ABAQUS. Based on the three-ball case, we change the initial velocity to get different launch velocities of the top ball as shown in Table 1. Since the height the top ball can reach is proportional to the launch velocity squared, we plot the relation between the launch velocity squared and the initial velocity in Fig. 5. If the energy dissipation rate is the same in each collision, we can expect the ratio of launch velocity is independent on the initial velocity, however, which is not the case shown in Fig. 5. Therefore, the energy dissipation rate is a function of mass and incident velocity of balls included in a collision, which indicates introducing energy dissipation rate α is an approximate treatment.

It should be noted that in these numerical cases the smallest value of the final height that the top ball can reach is about 6 times the corresponding original height, which shows the feature of the stacking ball system known as Galilean Cannon.

Table 1 Launch velocities for different initial velocities

Init Vel / Laun Vel	-1.0	-1.5	-2.0	-2.5	-3.0
Analytical	2.704	4.056	5.408	6.760	8.112
Numerical	3.013	4.110	5.057	6.566	7.302
Vel^2 Ratio τ	1.242	1.027	0.874	0.943	0.810

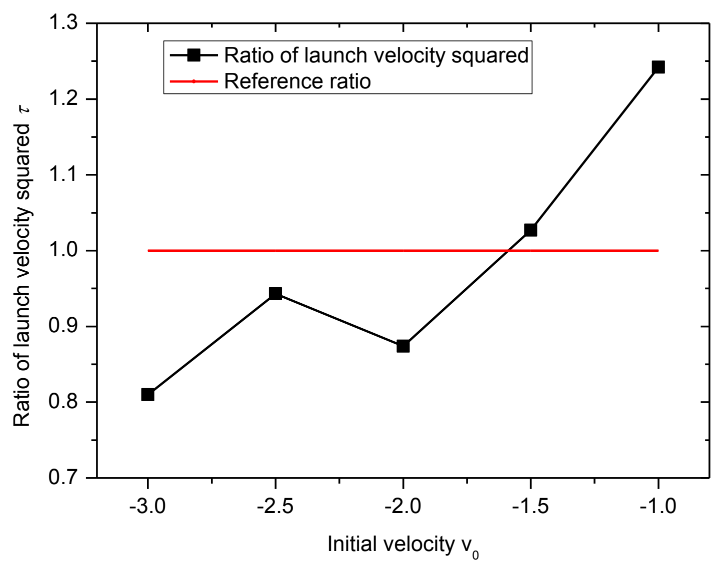


Fig. 5 The ratio of launch velocity squared

4 Optimization

Further study on this problem will focus on how to optimize the launch velocity of the top ball under certain condition. There are many factors that can influence the launch velocity, and the influence of initial velocity can be seen from Table 1. The launch velocity will monotonously change as the initial velocity, which can be an intuitive knowledge. However, how other factors affect the stacking ball system can be more complex. In this section, we will again use both theoretical and numerical tools to investigate this topic.

4.1 Optimization of rigid model

Since in our theoretical model described in Section2 has no relation with material parameters, the mass of balls can be a significant argument that will influence the launch velocity. Here we only study a simple three-ball case where the masses of the heaviest and lightest balls are fixed and the mass of the intermediate ball is variable. In the following we can prove there is an optimal mass of the intermediate ball to make the launch velocity largest.

Suppose one ball with mass m_1 and initial velocity v_{10} collides with another ball with mass m_2 and initial velocity v_{20}. After collision, the velocities of the two balls become v_1 and v_2, respectively. According to principles of linear momentum and kinematic energy conservation we have

$$m_1 v_{10} + m_2 v_{20} = m_1 v_1 + m_2 v_2, \tag{4.1}$$

$$\frac{1}{2} m_1 v_{10}^2 + \frac{1}{2} m_2 v_{20}^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2. \tag{4.2}$$

The solution to the above equations is

$$v_1 = \frac{(m_1 - m_2) v_{10} + 2 m_2 v_{20}}{m_1 + m_2}, \tag{4.3a}$$

$$v_2 = \frac{(m_2 - m_1) v_{20} + 2 m_1 v_{10}}{m_1 + m_2}. \tag{4.3b}$$

Here we consider this simplified three-ball case where the first ball with velocity v collides with the second ball with velocity $-v$, after which the second ball collides with the third ball with mass m_3 and velocity $-v$. Substituting $v_{10} = v$ and $v_{20} = -v$ into Eq. (4.3b), we can get

$$v_2 = \frac{3m_1 - m_2}{m_1 + m_2} v. \tag{4.4}$$

Then applying Eq. (4.3b) to the collision process of the second and third balls leads to the velocity of the third ball after collision

$$v_3 = \frac{-m_2^2 + (7m_1 - m_3) m_2 - m_1 m_3}{(m_1 + m_2)(m_2 + m_3)} v. \tag{4.5}$$

Now suppose m_1 and m_3 are fixed and we need to find such an m_2 to make v_3 to be maximum. Derive v_3 with respect to m_2 and we can get

$$\frac{\mathrm{d}v_3}{\mathrm{d}m_2} = \frac{-8 m_1 m_2^2 + 8 m_1^2 m_3}{(m_1 + m_2)^2 (m_2 + m_3)^2} v. \tag{4.6}$$

Let $\mathrm{d}v_3/\mathrm{d}m_2 = 0$ we can get

$$m_2 = \sqrt{m_1 m_3}. \tag{4.7}$$

We can further easily prove that $m_2 = \sqrt{m_1 m_3}$ is the optimal mass to make the launch velocity of the top ball largest and thus the final height largest.

4.2 Optimization of hyperelastic model

In this subsection, we will investigate how the constitutive coefficient, the relative thickness and the mass of balls influence the launch velocity by using finite element analysis and the hyperelastic Neo-Hookean model in ABAQUS.

Based on the three-ball system in Section 3, we change the coefficient C_{10} from 1.0×10^5 to 2.0×10^9 to see how the launch velocity varies. Fig. 6 shows the influence of constitutive coefficient C_{10} on the launch velocity of the top ball. We can see that when the material of balls is relatively soft, the launch velocity is not the smallest. As the material gets stiffer, the launch velocity will first decrease to a minimum value at $C_{10} = 1.0 \times 10^8$, and then increase to obtain larger launch velocities. The possible reason may be the combined action of energy dissipation inside the ball and momentum transmission between balls. The soft material has larger elastic deformation and thus can transmit linear momentum to the next ball better, while the stiff material has less energy dissipated inside the ball. However, the material with intermediate value of C_{10} has bad behavior on both sides.

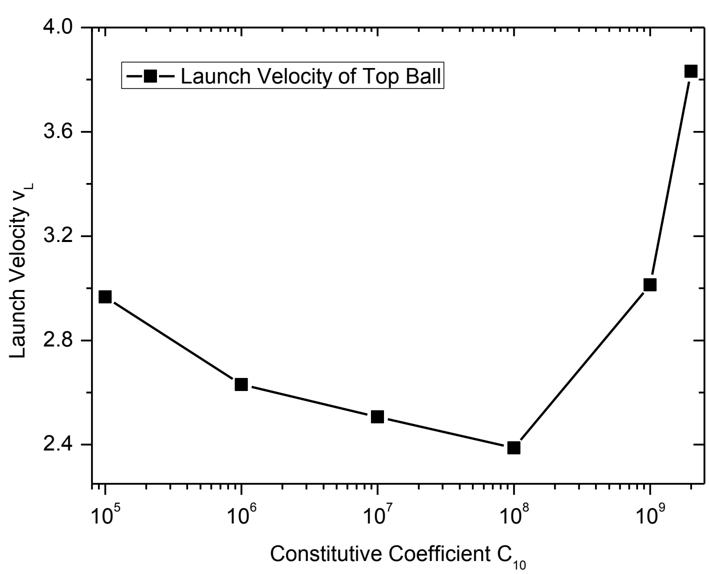


Fig. 6 The influence of coefficient C_{10} on launch velocity

Next the effect of relative thickness of spherical shells is studied. Here the relative thickness means the shell thickness normalized by radius of the ball, namely

$$\hat{d} = \frac{d_i}{r_i}, \tag{4.8}$$

where d_i and r_i are the thickness and radius of the i-th ball, respectively. From Fig. 7 we can see the launch velocity of the top ball has maximum value at $\hat{d} = 1.0$. This phenomenon can also be understood by the combined action of energy dissipation and momentum transmission as in the last case. However, the difference in this case is that the thin shells have less energy dissipation but are not good at transmitting momentum because they are too light. On the other hand, the thick shells have too much energy dissipated in the solid material. However, the shells with intermediate values can obtain a good balance on the two aspects and thus lead to larger launch velocities.

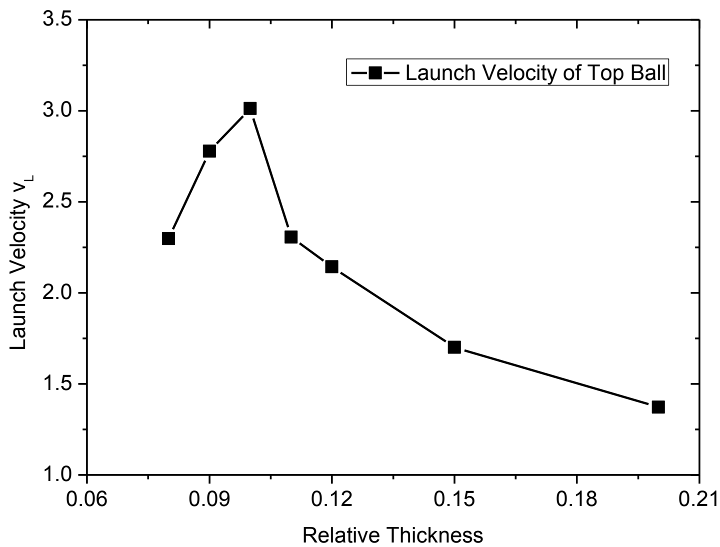


Fig. 7 The influence of relative thickness on launch velocity

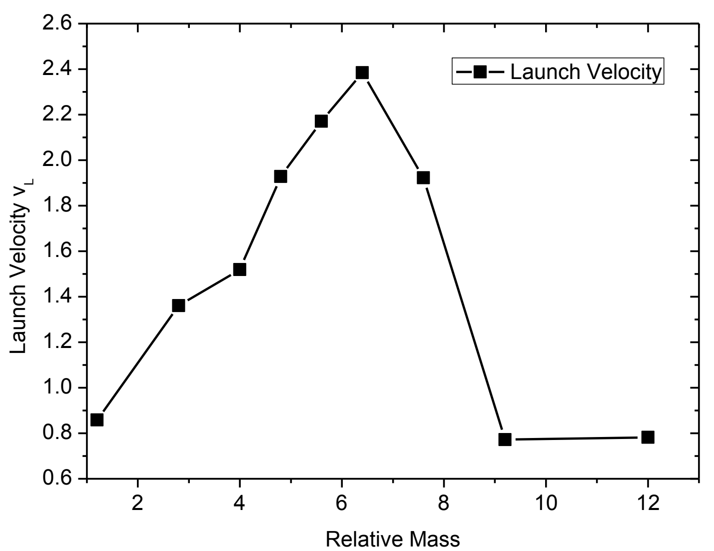


Fig. 8 The influence of relative mass of the middle ball on launch velocity

At last, we will use the numerical model to verify the optimization result given by the theoretical model in subsection 4.1 (i.e., Eq. (4.7)). Still based on the three-ball system, we can get the relation of launch velocity with relative mass of the middle ball as shown in Fig. 8. Here the relative mass of the middle ball is the mass of this ball normalized by the mass of the top ball. From Fig. 8 we can see there is really an optimal mass of the middle ball. Eq. (4.7) gives the optimal solution when the relative mass $\hat{m}_2 = 5.2$, while the peak in Fig. 8 corresponds to $\hat{m}_2 = 6.4$. Different physical models and analysis methods can account for this difference.

5 Concluding remarks

In this project, we use both theoretical model and finite element model to study the behavior of stacking ball system, usually called as Galilean Cannon, and relevant optimization problems associated with several key parameters including the coefficient in Neo-Hookean law, relative thickness of balls and relative mass of the middle ball. Our theoretical model and numerical model both exhibit the ability of the stacking ball system to launch a light ball to an extraordinary height, and reveal that this ability comes from transmission of kinematic energy and linear momentum from heavy balls to the light one. The results of relevant optimization problems indicate that the energy dissipation inside the ball and momentum transmission between balls can be two significant factors that influence the launch velocity of the top ball. In addition, for a three-ball system, the theoretical model gives the optimal mass of the middle ball as the square root of the product of masses of the other two balls.

References

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