EN 4 HW #6 SOLUTIONS

7 pts 1. a) The cantilever acts like a spring with spring constant $k = 3EI/L^3$. The frequency of vibration for a mass m_{eff} attached to this spring is then

1 pt.
$$\omega_n = \sqrt{\frac{k}{m_{eff}}} = \sqrt{\frac{3EI}{L^3 m_{eff}}}$$

We choose m_{eff} so this frequency matches the true frequency $\omega_n = 1.03\sqrt{Ewt^3/\ell^3 m}$ where $\ell = L$. So

2 pts.
$$(1.03)^2 \frac{3EI}{L^3 m_{eff}} = \frac{Ewt^3}{L^3 m} \Rightarrow \frac{m_{eff}}{m} = \frac{(1.03)^2 3I}{wt^3}$$

For a rectangular beam, $I = \frac{wt^3}{12}$ so $\frac{m_{eff}}{m} = \frac{(1.03)^{-2} 3}{12} = 0.235$

b) Upon adding the E-coli, of mass m_{ec}, the natural frequency becomes

2 pts.
$$\omega_n' = \sqrt{\frac{3EI}{L^3(m_{eff}+m_{ec})}} = \sqrt{\frac{3EI}{L^3 m_{eff}}}\frac{1}{\sqrt{1+m_{ec}/m_{eff}}}$$

or

$$\omega_n' = \omega_n \frac{1}{(1+m_{ce}/m_{eff})^{1/2}}$$

The change in frequency is then

$$\Delta\omega_n = \omega_n' - \omega_n = \omega_n\left(\frac{1}{(1+m_{ce}/m_{eff})^{1/2}} - 1\right)$$

If $m_{ce}/m_{eff} \ll 1$, then
$$(1+m_{ce}/m_{eff})^{-1/2} \sim 1 - \frac{1}{2}\frac{m_{ce}}{m_{eff}}$$

So

$$\Delta\omega_n = \omega_n\left(\frac{1}{2}\frac{m_{ce}}{m_{eff}}\right) = \omega_n\left(\frac{1}{2}\frac{m}{m_{eff}}\frac{m_{ec}}{m}\right) = 2.12\,\omega_n\left(\frac{m_{ec}}{m}\right)$$

in terms of the original natural frequency and the true mass m.

$\rightarrow m = 2.72\times10^{-13}\,kg$ (2)

ω_n for the beam shown is, using $m=(wLt)\rho$,

$$\omega_n = 1.03\sqrt{\frac{Ewt^3}{L^3 wLt\rho}} = 1.03\sqrt{\frac{Et^2}{L^4\rho}} = 1.03\sqrt{\frac{110\times10^9\,N/m^2\,(320\times10^{-9}\,m)^2}{(25\times10^{-6}\,m)^4\,3.4\times10^{-3}\,kg/(10^{-2}\,m)^3}}$$

or

$$\omega_n = 2.9\times10^6\,\frac{rad}{s} \Rightarrow f_n = 4.63\times10^5\,Hz = 463\,kHz$$

From the middle figure showing data vs. # of E-coli cells, we can estimate $\Delta f_n = 3\,kHz$. So,

$$\Delta f_n = (3\,kHz) = (463\,kHz)(2.12)(M_{ec}/m)$$

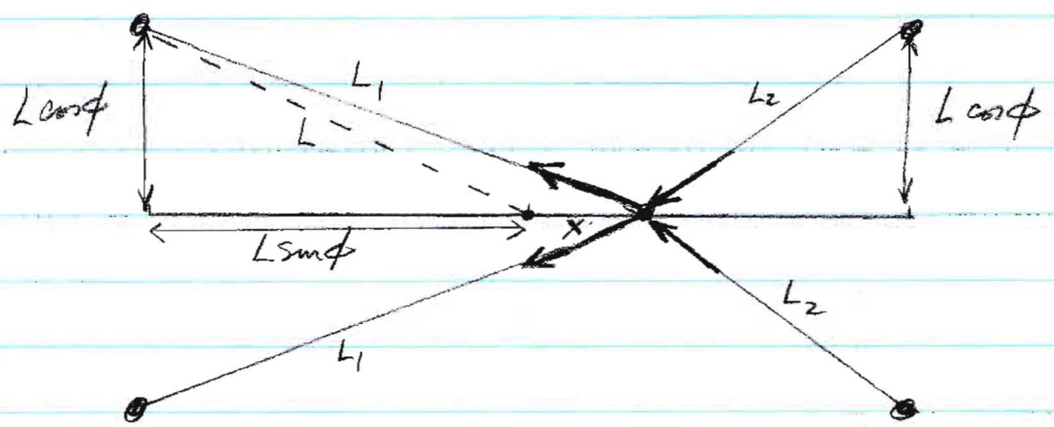
2 pts. $\Rightarrow M_{ec}/m = 3.05\times10^{-3} \Rightarrow M_{ec} = 3.05\times10^{-3}\,m$

$$\underline{\underline{M_{ec} = 8.31\times10^{-16}\,kg}}$$

12 pts. 2. At equilibrium, the forces due to the four springs all cancel. If the mass is displaced by x then the spring forces change by k(change in spring length) and the angle changes slightly as well. The initial length does matter, so we take L to be the unstretched length. To get ω_n, we need to draw a F.B.D. and determine the EOM.

F.B.D. (exaggerated)

3 pts



The y forces all balance. Summing the forces in the x direction we obtain

3 pts.

$$\sum F_x = \underbrace{-2k(L_1 - L)}_{\text{magnitude}} \underbrace{\frac{L\sin\phi + x}{L_1}}_{x\text{ component}} + 2k(L_2 - L)\left(\frac{L\sin\phi - x}{L_2}\right)$$

By geometry,

$$L_1 = \left((L\sin\phi + x)^2 + (L\cos\phi)^2\right)^{1/2} = \left(L^2 + 2L\sin\phi\, x + x^2\right)^{1/2}$$
$$= L\left(1 + 2\sin\phi \frac{x}{L} + x^2/L^2\right)^{1/2}$$
$$\simeq L\left(1 + \sin\phi\, x/L\right) \qquad x/L \ll 1$$
$$= L + x\sin\phi$$

and

$$L_2 = \left((L\sin\phi - x)^2 + (L\cos\phi)^2\right)^{1/2} = \left(L^2 - 2L\sin\phi\, x + x^2\right)^{1/2}$$
$$= L\left(1 - 2\sin\phi\, x/L + x^2/L^2\right)^{1/2}$$
$$\simeq L\left(1 - \sin\phi\, x/L\right) = L - x\sin\phi$$

2 pts So, $L_1 - L \approx x\sin\phi$, $L_2 - L \approx -x\sin\phi$

Plugging in:

$$\sum F_x = -2k\sin\phi\, x \underbrace{\left(\frac{L\sin\phi + x}{L + x\sin\phi}\right)}_{=\frac{L(\sin\phi + x/L)}{L(1+\sin\phi\, x/L)} \approx \sin\phi \quad x/L \ll 1} - 2k\sin\phi\, x \underbrace{\left(\frac{L\sin\phi - x}{L - x\sin\phi}\right)}_{=\frac{L(\sin\phi - x)}{L(1-\sin\phi\, x/L)} \approx \sin\phi \quad x/L \ll 1}$$

→ terms ~ x/L will be multiplied by x in the force and so give terms ~ x^2 and all all small.

2 pts. So

$$\sum F_x \approx -2k\sin\phi\, x\,(\sin\phi) - 2k\sin\phi\, x\,(\sin\phi)$$

$$\boxed{\sum F_x = -4k\sin^2\phi\, x}$$

Makes sense. At $\phi = 90$, we get $4k$ – all four springs act together. At $\phi = 0$, we get 0 – springs don't contribute any force at order x/L. (see problem 3 if $T=0$, also!)

$$\Rightarrow \underset{\sim}{F} = m\underset{\sim}{a} \Rightarrow m\frac{d^2x}{dt^2} = -4k\sin^2\phi\, x$$

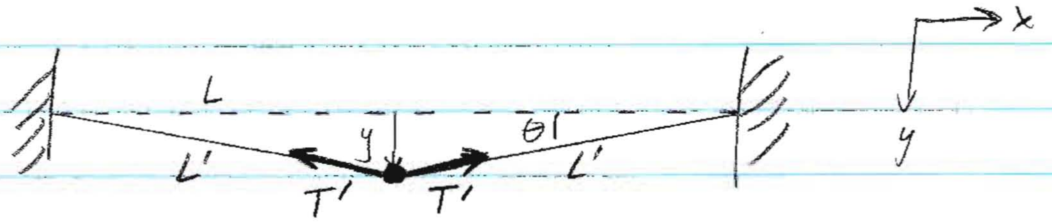
$$\Rightarrow \frac{d^2x}{dt^2} + \frac{4k\sin^2\phi}{m}x = 0$$

2 pts.

$$\Rightarrow \omega_n = \sqrt{\frac{4k\sin^2\phi}{m}}$$

10 pts **3.** As in Prob. 2, draw F.B.D and get EOM

3 pts



Tension T' is unknown at the moment. When $y=0$, tension is T. For $y \neq 0$, the tension T' will change due to the change in length of the wire. However, that change in length is quite small, so that $T' = T$ is accurate for $y/L \ll 1$.

Let's see this explicitly; from geometry we have

2 pts.

$$L' = (L^2 + y^2)^{1/2} = L(1 + y^2/L^2)^{1/2} \sim L\left(1 + \tfrac{1}{2} y^2/L^2\right)$$

so

$$\Delta L = L' - L = \tfrac{1}{2} y^2/L$$

The tension, assuming the wires act like linear springs is then

$$T' = T + \underbrace{k\Delta L}_{\text{extra tension due to extra stretch}}$$

So,

2 pts.

$$\sum F_y = -2T' \sin\theta = -2T'\left(\frac{y}{L'}\right)$$

$$= -2(T + k\Delta L)\,\frac{y}{L + \Delta L}$$

$$= -2\left(T + \tfrac{1}{2} k y^2/L\right)\frac{y}{L\left(1 + \tfrac{1}{2} y^2/L^2\right)}$$

Since there is a factor of y just from $\sin\theta$, all other terms with y^2 are higher order and can be neglected for small y.

Thus,

2 pts.

$$\sum F_y \approx -2Ty/L \qquad k \text{ does not appear!}$$

Then

$$\underset{\sim}{F} = m\underset{\sim}{a} \Rightarrow m\frac{d^2y}{dt^2} + \frac{2T}{L} y = 0 \Rightarrow \frac{d^2y}{dt^2} + \frac{2T}{mL} y = 0$$

1 pt. and so $\boxed{\omega_n = \sqrt{\frac{2T}{mL}}}$

For a frequency of $f = 110$ Hz, with $T = 750N$ and $L = 1m$, we have

$$2\pi(110)\frac{1}{s} = \sqrt{\frac{2(750N)}{m(1m)}}$$

$$\Rightarrow \quad m = \frac{1500\ kg\,m/s^2}{(2\pi)^2(110)^2\frac{1}{s^2}\ 1m} = .0031kg = \underline{\underline{3.1\ grams}}$$

6 pts 4. τ_d: The time between peaks in the amplitude is 0.1 s

2 pts so $\underline{\underline{\tau_d = 0.1s}}$

ζ : The peak heights at two successive peaks give the "log decrement" δ, which is related to ζ as

2 pts

$$\zeta = \frac{\delta}{\sqrt{\delta^2 + (2\pi)^2}}$$

Here, $\delta = \ln\left(\frac{.097}{.027}\right) = 1.28$ so $\underline{\underline{\zeta = 0.2}}$

2 pts.

ω_n : $\tau_d = \frac{2\pi}{\omega_d}$; $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$ so

$$\omega_n = \frac{2\pi}{\tau_d\sqrt{1-\zeta^2}} = \underline{\underline{64.1\ \frac{rad}{s}}}$$