EN4 HW#7 SOLUTIONS

1. The EOM derived for the Spinal Pendulum was (see prior solutions)

$$ml^2 \frac{d^2\theta}{dt^2} + \eta \frac{d\theta}{dt} + k\theta + mgl\sin\theta = 0$$

For small angles (vibration problems), we approximate $\sin\theta \approx \theta$ to obtain

$$ml^2 \frac{d^2\theta}{dt^2} + \eta \frac{d\theta}{dt} + (k+mgl)\,\theta = 0$$

which is in the form of a damped vibration problem

$$M\ddot{x} + c\dot{x} + kx = 0$$

Dividing by ml^2, we get

$$\ddot{\theta} + \frac{\eta}{ml^2}\dot{\theta} + \frac{k+mgl}{ml^2}\theta = 0$$

The natural frequency is then

$$\omega_n = \sqrt{\frac{k+mgl}{ml^2}}$$

The damping coefficient is

$$2\zeta\omega_n = \frac{\eta}{ml^2} \Rightarrow \zeta = \frac{\eta}{4ml^2\omega_n}$$

Plugging in values: $l = 0.46\,m$, $m = 50\,kg$, $k = 200\,\frac{Nm}{rad}$, $\eta = 4\,Nms/rad$ we find

$$\underline{\underline{\omega_n = 6.34\,\tfrac{rad}{s}}}\,,\quad \underline{\underline{\zeta = 0.03}} \Rightarrow \underline{\underline{\tau_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.998\,sec}}$$

From MATLAB results, look at peak difference between $t=0$ and the 6^{th} peak (larger difference and so easier to analyze):

$$\delta = \ln\frac{\theta_0}{\theta_{6^{th}peak}} = \ln\frac{.0277}{.0088} = 1.144 = 6\zeta\omega_n\tau_d = 6\cdot\frac{2\pi\zeta}{(1-\zeta^2)^{1/2}}$$

so

$$\zeta = \frac{\delta/6}{\sqrt{(\delta/6)^2 + (2\pi)^2}} = \underline{\underline{.03}}$$ agrees with analytic value above

2. During operation at constant T, the thruster + stand compresses the spring to a distance satisfying

$$T - kx_0 = 0 \quad \text{or} \quad x_0 = T/k$$

This x_0 is the initial position of the system when the thrust is turned off. The velocity at this instant is zero.

To have $x_0 < 1$ ft requires $T/k < 1$ ft $\Rightarrow$ $\underline{\underline{k > 10{,}000\ \text{lb/ft}}}$

To return to equilibrium as fast as possible requires a design for which $\zeta = 1$ (Critical damping). For this spring, mass, dashpot system we have

$$\zeta = \frac{c}{2m\omega_n}, \quad \omega_n = \sqrt{k/m}$$

For $k = 10000$ lb/ft, $mg = 1500$ lb, we have $\underline{\underline{\omega_n = 14.65\ \tfrac{\text{rad}}{\text{s}}}}$

So $\zeta = 1$ requires $c = 2m\omega_n = 2\,\dfrac{(1500\ \text{lb})}{32.2\ \text{ft/s}^2}\, 14.65\ \dfrac{\text{rad}}{\text{s}} = \underline{\underline{1365\ \text{lb-ft/s}}}$

The motion is then described by

$$x(t) = (C_1 + C_2 t)\, e^{-\omega_n t}$$

Applying initial conditions,

$$x(0) = \underline{\underline{C_1}} = x_0$$

$$\dot{x}(0) = \left(-\omega_n C_1 e^{-\omega_n t} + C_2 e^{-\omega_n t} - \omega_n C_2 \underbrace{t}_{0} e^{-\omega_n t}\right)\Big|_{t=0} = 0$$

so

$$-\omega_n C_1 + C_2 = 0 \Rightarrow C_2 = \omega_n C_1 = \omega_n x_0$$

So,

$$x(t) = x_0 (1 + \omega_n t)\, e^{-\omega_n t}$$

3. The seismograph is a simple spring, mass, dashpot system so that the system parameters are

$$\omega_n = \sqrt{\frac{k}{m}} \quad \text{and} \quad \zeta = \frac{c}{2m\omega_n}$$

The steady-state motion under "base excitation", i.e. when driven by an earthquake at frequency ω, was solved generally in class. The amplitude of vibration relative to the base itself is given by

$$\frac{Z_1}{Y} = \left(\frac{\omega}{\omega_n}\right)^2 M \quad \text{with} \quad M = \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

To detect earthquakes at $f = 3\ \text{Hz}$, we should design a system with a resonance at $3\ \text{Hz}$, i.e.

$$\omega_n = 2\pi \cdot 3\ \frac{\text{rad}}{s} = 6\pi\ \frac{\text{rad}}{s}$$

At resonance, $\omega/\omega_n = 1$, we have $\frac{Z}{Y} = \frac{1}{2\zeta}$

To detect amplitudes of 0.01 mm when the background vibrations are of amplitude 0.1 mm requires $Z/Y > \frac{0.1\text{mm}}{0.01\text{mm}} = 10$

so

$$\frac{1}{2\zeta} > 10 \Rightarrow \underline{\zeta < 0.05}$$

However, we also want $Z/Y < \frac{30\text{mm}}{1\text{mm}} = 30 \Rightarrow \underline{\zeta > 0.0166}$

Pushing the system to this limit, we want

$$\omega_n = 6\pi\ \text{rad/s} \quad ; \quad \zeta = 0.0166$$

Any combo of k, m, c achieving this is acceptable.

If $\underline{m = 1\ \text{kg}}$ then $\underline{k = 355.3\ \text{N/m}}$ and $c = .0166 \cdot 2 \cdot 1\text{kg} \cdot 6\pi/s = \underline{.625\ \text{N-s/m}}$

4. From last week, we have

$$\omega_n = 64.1 \frac{rad}{s}, \quad \zeta = 0.2$$

For a mass imbalance in a rotating machine, the amplitude of vibration is given by

$$X = \frac{\Delta m e}{m}\left(\frac{\omega}{\omega_n}\right)^2 M$$

where M is the standard magnification factor.

Here, we measure $X = 20 \times 10^{-6}$ m at 6000 rpm.

a) For 6000 rpm, $\omega = 6000 \times 2\pi / 60s = 200\pi \frac{rad}{s}$

so $\frac{\omega}{\omega_n} = 9.802$

Solving for $\Delta m e$,

$$\Delta m e = \frac{m X}{\left(\frac{\omega}{\omega_n}\right)^2 M} = \frac{450 kg \times 20 \times 10^{-6} m}{96.08 \; M} = \underline{\underline{.009 \text{ kg-m}}}$$

b) The maximum force is (see notes)

$$F_T = \Delta m e \frac{k}{m} \sqrt{1 + \left(2\zeta\left(\frac{\omega}{\omega_n}\right)\right)^2} \left(\frac{\omega}{\omega_n}\right)^2 M$$

or

$$F_T = \Delta m e \, \omega_n^2 \sqrt{1 + \left(2\zeta\left(\frac{\omega}{\omega_n}\right)\right)^2} \left(\frac{\omega}{\omega_n}\right)^2 M$$

$$= .009 \text{ kg-m} \cdot 64.1^2 \frac{1}{s^2} \; (4.04) \; (1.01) = \underline{\underline{150.9 \text{ N}}}$$

c) If we double c, we double $\zeta \Rightarrow \zeta = 0.4$.

Then,

$$F_T = .009 \text{ kg-m } 64.1^2 \frac{1}{s^2} \; (7.96) = \underline{294.3 \text{ N}}$$

Increased due to damping.

d) At ω_n instead, $\zeta = 0.2$, $\underline{F_T = 99.57 \text{ N}}$

Lower, even though system is at resonance!!

5. 2-Story Building

From the on-line notes, the EOMs were shown to be

(1) $$\frac{d^2 l_1}{dt^2} = -g - 2\frac{k}{m}(l_1 - L) + 2\frac{k}{m}(l_2 - L) + \omega^2 d_0 \sin \omega t$$

(2) $$\frac{d^2 l_1}{dt^2} + \frac{d^2 l_2}{dt^2} = -g - 2\frac{k}{m}(l_2 - L) + \omega^2 d_0 \sin \omega t$$

We are not interested in the driving force, so set $d_0 = 0$.
The second equation is not just for l_2, so combine eqns (1) and (2) (subtract eq (1) from eq. (2)) to get the pair

$$\frac{d^2 l_1}{dt^2} = -g - 2\frac{k}{m}(l_1 - L) + 2\frac{k}{m}(l_2 - L)$$

$$\frac{d^2 l_2}{dt^2} = -\frac{2k}{m}(l_2 - l_1) - 2\frac{k}{m}(l_2 - L)$$

Now neglect constants (redefines l_1, l_2 in terms of displacements from equilibrium) and get

$$\frac{d^2 l_1}{dt^2} = -\frac{2k}{m} l_1 + \frac{2k}{m} l_2$$

$$\frac{d^2 l_2}{dt^2} = \frac{2k}{m} l_1 - \frac{4k}{m} l_2$$

or

$$\frac{d^2}{dt^2}\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} -2k/m & 2k/m \\ 2k/m & -4k/m \end{bmatrix}\begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$$

so we have identified the coefficients of the matrix M.

Solve the eigenvalue problem:

$$\det\begin{vmatrix} -2\frac{k}{m}-\lambda^2 & 2k/m \\ 2k/m & -4k/m-\lambda^2 \end{vmatrix} = \left(-\frac{2k}{m}-\lambda^2\right)\left(-\frac{4k}{m}-\lambda^2\right)-\left(2\frac{k}{m}\right)^2 = 0$$

Expanding out the polynomial, we have:

$$8\left(\frac{k}{m}\right)^2+6\left(\frac{k}{m}\right)\lambda^2+\lambda^4-4\left(\frac{k}{m}\right)^2 = \lambda^4+6\left(\frac{k}{m}\right)\lambda^2+4\left(\frac{k}{m}\right)^2=0$$

This is a quadratic equation for λ^2, so the solution is

$$\lambda^2 = \frac{-6\left(\frac{k}{m}\right)\pm\left(36(k/m)^2-16(k/m)^2\right)^{1/2}}{2}$$

or

$$\lambda^2 = \left(-3\pm\sqrt{5}\right)\left(\frac{k}{m}\right) \Rightarrow \lambda = \pm i(0.877)\sqrt{\frac{k}{m}},\ \pm i\left(2.29\sqrt{\frac{k}{m}}\right)$$

so there are two natural frequencies:

$$\omega_n^{(1)} = 0.877\sqrt{\frac{k}{m}} \quad\text{and}\quad \omega_n^{(2)} = 2.29\sqrt{\frac{k}{m}}$$

For $k = 100\ N/m$, $m = 1\ kg$ (values used in prior matlab studies)

$$\omega_n^{(1)} = 8.77\ \frac{rad}{s} \quad ; \quad \omega_n^{(2)} = 22.9\ \frac{rad}{s}$$

Thus, the responses computed by the MATLAB code at $\omega = 9\ \frac{rad}{s}$ and $\omega = 23\ rad/s$ are very close to

Resonance $(\omega = \omega_n)$ and the vibration amplitudes become very large when driven by $d_0 \sin\omega t$ with $d_0 = 0.1m$.
(see calculations in lecture notes)