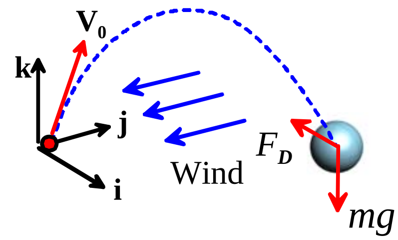
Division of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 3: Solving equations of motion for particles
Due Friday Feb 13



1. A spherical projectile with diameter D and mass m is launched from the origin with initial velocity vector $\mathbf{V}_0 = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}$. A surface wind blows with air velocity $\mathbf{V}_w = W_x\mathbf{i} + W_y\mathbf{j} + W_z\mathbf{k}$, subjecting the projectile to a drag force

$$\mathbf{F}_D = -\frac{1}{2}\rho C_D \frac{\pi D^2}{4} V\left((v_x - W_x)\mathbf{i} + (v_y - W_y)\mathbf{j} + (v_z - W_z)\mathbf{k} \right)$$

where $V = \sqrt{(v_x - W_x)^2 + (v_y - W_y)^2 + (v_z - W_z)^2}$ is the magnitude of the particle's velocity relative to the air, $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$ is the projectile velocity, C_D is the drag coefficient, and ρ is the air density.

a. The motion of the system will be described using the (x,y,z) coordinates of the particle. Write down the acceleration vector in terms of time derivatives of these variables.

b. Write down the vector equation of motion for the particle (**F**=m**a**).

c. Show that the equation of motion can be expressed in MATLAB form as

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ -cV(v_x - W_x) \\ -cV(v_y - W_y) \\ -g - cV(v_z - W_z) \end{bmatrix}$$

and give a formula for the constant c.

d. Modify the MATLAB code discussed in class to calculate and plot the trajectory. (THERE IS NO NEED TO SUBMIT A SOLUTION TO THIS PROBLEM)

e. Test your code by calculating the trajectory of a projectile that is launched with velocity $\mathbf{V}_0 = 13\mathbf{i} + 25\mathbf{k}$ m/s, with $c=0$, and compare the MATLAB solution with the exact solution. A good way to do this is to plot the distance between the points where the particle should be, and where MATLAB predicts it to be, as a function of time.

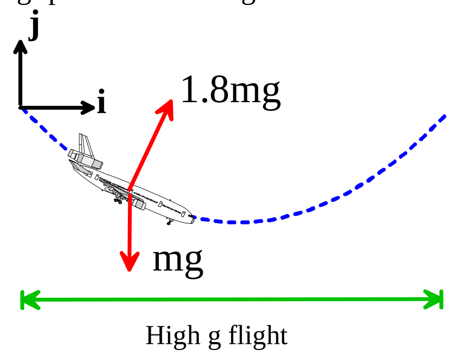
f. Finally, run simulations with parameters representing a golf-ball (m=0.0459kg, $C_D = 0.25$, D=4.115cm, air density 1.02 kg/m^3) launched at 67 m/s (150 mph) at an angle of 35 degrees to the horizontal. Calculate the difference in length of a drive hit into a 6.7m/s (15 mph) head wind to that hit into a 6.7m/s tailwind. Also, calculate the lateral deflection caused by a 6.7 m/s side-wind.

There is no need to hand in MATLAB code or graphs for this problem – just report your values.

2. In this problem we revisit the NASA 'zero-g simulator' aircraft trajectory. In the preceding homework you analyzed the `zero g' portion of the flight-path. The goal here is to calculate the velocity and trajectory of the aircraft as it pulls out of the dive during the 'high-g' portion of the flight.

The figure shows a free body diagram for the aircraft. No engine power is used during this phase of the flight (can you see why?) and drag has been neglected for simplicity. Note that:

- The lift force acts perpendicular to the path of the aircraft.
- The aircraft is flown to keep the magnitude of the lift force constant (this is to avoid structural damage to the airframe). The magnitude is 1.8*mg* where *m* is the aircraft mass and *g* is the gravitational acceleration.



a. The motion of the aircraft will be described by the components of its position vector (*x*,*y*). Write down the acceleration vector in terms of time derivatives of these variables.

b. Let $\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j}$ denote the velocity vector of the aircraft. Noting that the velocity vector is tangent to the path, find a formula for a unit vector *perpendicular* to the path (and parallel to the lift force), in terms of v_x and v_y. Use the result to write down the resultant force vector acting on the aircraft.

c. Write down the vector equation of motion for the aircraft **F**=m**a**.

d. Hence, show that the equation of motion for the aircraft can be expressed in the MATLAB form

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ -1.8 g v_y / V \\ -g + 1.8 g v_x / V \end{bmatrix}$$

where $V = \sqrt{v_x^2 + v_y^2}$.

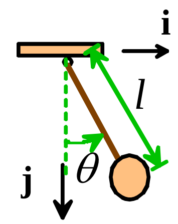
e. Write a MATLAB script that will solve this equation of motion, given initial values for [x,y, v_x ,v_y].

f. Assume that the aircraft enters the 'high-g' portion of flight with velocity $\mathbf{v} = 244\mathbf{i} - 264\mathbf{j}$ knots (CONVERT TO m/s IF YOU ARE USING SI UNITS FOR g!!). You can assume that the aircraft is at the origin at this instant. Plot (i) the subsequent trajectory of the aircraft; and (ii) the magnitude of the velocity of the aircraft as a function of time, for a period of 50 seconds. What is the maximum speed of the aircraft during the dive?

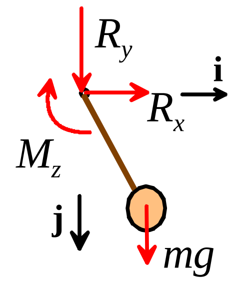
This calculation suggests that the aircraft is being flown very close to its design limits. The max airspeed for a 737 is listed as Mach 0.84, which is 251 m/s at 30000ft. Our calculation predicts that this speed is exceeded. We have neglected drag, so it is likely that the aircraft just hits its max allowable speed at the bottom of the dive. It is this limitation, together with the 1.8mg lift force limitation, that restrict the maximum time for zero g flight. NASA have clearly done their homework…

3. To understand the motivation for this problem, read through the following publication:

> Crisco Joseph J; Fujita Lindsey; Spenciner David B, 'The dynamic flexion/extension properties of the lumbar spine in vitro using a novel pendulum system.' Journal of biomechanics 2007; 40(12):2767-73



The goal of this problem is to analyze the motion of the pendulum system used in Professor Crisco's experiments. You will calculate the swing angle of the pendulum θ shown in the figure as a function of time.



The figure shows a free body diagram for the pendulum. The shaft of the pendulum is treated as a massless frame. The moment acting at the pivot represents the effects of elastic stiffness and viscous damping in the spinal joint. The magnitude of the moment is related to the rate of swing of the pendulum by

$$M_z = \eta\omega + k\theta$$

where η, k are constants and $\omega = d\theta / dt$ is the rotation rate of the shaft

a. Write down the position vector of the mass on the end of the pendulum in terms of θ.

b. Hence, calculate the acceleration vector of the mass on the end of the pendulum, in terms of *l*, θ and its time derivatives.

c. Hence, write down **F**=m**a** for the system.

d. Write down **M**=**0** for the pendulum (the shaft and mass together). Remember to take moments about the center of mass of the system (i.e. the particle)

e. Eliminate the unknown reactions R_x, R_y from the equations of motion in (c) and (d) (you can solve the two components of (c) for the reactions and substitute into (d), then simplify the result with trig formulas) to show that θ satisfies the equation

$$ml^2 \frac{d^2\theta}{dt^2} + \eta \frac{d\theta}{dt} + k\theta + mgl\sin\theta = 0$$

f. Hence, show that the equations of motion can be expressed in MATLAB form as

$$\frac{d}{dt}\begin{bmatrix}\theta \\ \omega\end{bmatrix} = \begin{bmatrix}\omega \\ \frac{-\eta}{ml^2}\omega - \frac{k}{ml^2}\theta - \frac{g}{l}\sin\theta\end{bmatrix}$$

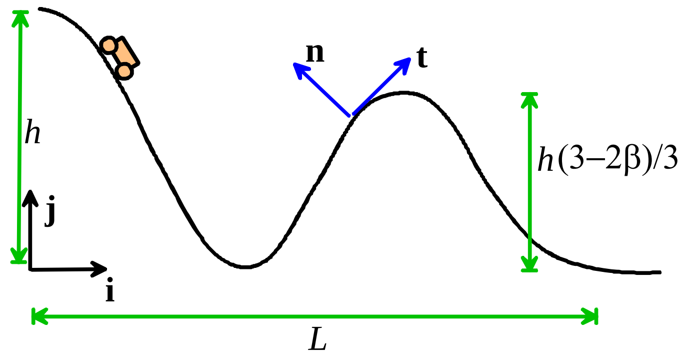
g. Write a MATLAB script to integrate the equations of motion. Check your code by running a simulation with $\eta = k = 0$, *l=g* and initial conditions $\theta = 0.05$ radians, $\omega = 0$ at time *t=0*. Plot θ as a function of time (a 25 sec time interval is good) and compare the solution with the approximate result $\theta = 0.05\cos(t\sqrt{g / l})$ for small amplitude oscillation of a pendulum.

h. Finally, run a simulation with parameters similar to those used in the experiments: $l = 46cm$, *m*=50kg, *k*=200Nm/radian, $\eta = 4Nms / radian$. Use initial conditions $\theta = 0.0277$ radians and $\omega = 0$, and plot the angle of the pendulum as a function of time. Compare the predicted period of oscillation with the value of 1 second measured experimentally.

4. A roller-coaster ride has profile given by

$$y = h(1-\beta\frac{x}{L})\cos^2\frac{3\pi x}{2L}$$

as shown in the picture, where $0<\beta<1$ is a constant that controls the height of the second peak. The ride must be designed so that cars will travel over the second peak in the ride.



The objective of this problem is to predict the motion of a car as it travels along the path, accounting for the effects of air resistance. The calculation can then be used to estimate a value for β.

The magnitude of the air drag force can be calculated as $F_D = \frac{1}{2}\rho C_D A V^2$, where ρ is air density, C_D is the drag coefficient; *A* is the frontal area of the car. The car has mass *m*.

a. In solving this problem it will be helpful to introduce the *slope* μ and *bending* κ of the path, defined as

$$\mu = \frac{dy}{dx} \qquad \kappa = \frac{d^2y}{dx^2}$$

Calculate formulas for μ and κ in terms of *x* (use MAPLE to do the calculus, and use the command combine(*enter expression to be simplified here*, trig) to simplify the results).

b. The motion of the car will be described using its (*x*,*y*) coordinates. Write down the acceleration vector in terms of these coordinates.

c. Draw a free body diagram for the car. Include (i) A reaction force *N* exerted on the car by the rollercoaster; (ii) An air drag force and (iii) gravity.

d. Note that

$$\mathbf{t} = \frac{(\mathbf{i}+\mu\mathbf{j})}{\sqrt{1+\mu^2}}$$

is a unit vector tangent to the path (can you see this?). Hence, find a unit vector **n** normal to the path. Use these results to write down the resultant force vector acting on the car, in terms of the slope μ and other variables.

e. Write down the vector equation of motion for the car **F**=m**a**. You can leave the expression in terms of F_D.

f. A constraint equation is required to ensure that the car follows the path of the rollercoaster. Note that *y* is related to *x* by the roller-coaster profile. Hence, show that the accelerations in the horizontal and vertical direction must be related by

$$\frac{d^2y}{dt^2} = \mu \frac{d^2x}{dt^2} + \kappa\left(\frac{dx}{dt}\right)^2$$

(This is tricky – start by showing $dy/dt = \mu dx/dt$ using the chain rule, and then differentiate again).

g. Show that the equations of motion in (e) and the constraint equation in (f) can be expressed in the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dfrac{\mu}{m\sqrt{1+\mu^2}} \\ 0 & 0 & 0 & 1 & \dfrac{-1}{m\sqrt{1+\mu^2}} \\ 0 & 0 & \mu & -1 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \\ \dfrac{dv_x}{dt} \\ \dfrac{dv_y}{dt} \\ N \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ \dfrac{-cV^2}{\sqrt{1+\mu^2}} \\ \dfrac{-\mu c V^2}{\sqrt{1+\mu^2}} - g \\ -\kappa v_x^2 \end{bmatrix}$$

where *V* is the magnitude of the car's velocity, and give a formula for the constant c in terms of air density ρ, the drag coefficient of the car C_D; its frontal area *A* and its mass *m*.

h. Write a MATLAB function to integrate the equations of motion. To test your code, run the simulation with $\beta = c = 0$, and set $h=L=m=1$. Start the car running with initial position $x=0$, $y=h$, and initial velocity (0.01,0), and run the simulation for time period [0,10]. Plot the trajectory of the car (use plot(y(:,1),y(:,2)). Note that the exact trajectory should be $y = \cos^2 \frac{3\pi x}{2}$. You should find that the trajectory is close to, but not exactly equal to this solution – if it is not close, you messed up your code (if this happens check your formulas for μ and κ in the code carefully). There is no need to submit your MATLAB code, but please hand in a copy of your graph.

i. Fix the error by increasing the accuracy of the MATLAB computation. Hand in a copy of the same graph you plotted in (h) but with better accuracy.

j. The ride is supposed to end when $x=L$ – add an 'event' function to your MATLAB code that will stop the computation when the car reaches $x=L$. There is no need to hand in a solution to this problem.

k. Finally, run your simulation with the following parameters:
   - Height h = 100m; length L=500m
   - Car mass 500kg
   - Car frontal area $4m^2$
   - Drag coefficient 0.1
   - Air density $1.02\, kg\ m^{-3}$

(you will also need to increase the simulation time) and use the simulations to estimate the value of β required for the car to get over the peak at $x=3L/4$. (you can do this by trial and error). Suggest a suitable value of β, and hand in a plot showing the speed of the car as a function of x. Also, calculate how long the ride will last.