EN40: Dynamics and Vibrations

Homework 5: Linear and Angular Momentum
This homework will not be graded

Division of Engineering
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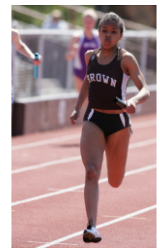
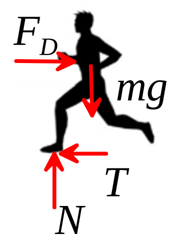
1. In a representative stride, a runner has a foot in contact with the ground for 0.1sec and is airborne for 0.2 sec. (see, e.g. Weyand *et al* 'Ambulatory estimates of maximal aerobic power from foot-ground contact times and heart rates in running humans,' *J. Appl. Physiol*, **91**, 451-458 (2001). Use the impulse-momentum equation to estimate the average vertical contact force that acts on your feet during the time that they are in contact with the ground.



The linear momentum of the runner is $mv\mathbf{i}$ where v is her speed, and m is her mass. We can idealize the runner as a particle. The impulse-momentum equation states that the change in linear momentum $\Delta\mathbf{p}$ during a time interval Δt is equal to the total impulse exerted by the forces acting on the particle



$$\int_0^{\Delta t} \mathbf{F} dt = \Delta\mathbf{p}$$

Since the linear momentum is constant, we have that

$$\int_0^{\Delta t} \mathbf{F} dt = \mathbf{0}$$

A free body diagram for the runner is shown in the figure. Note that $N=T=0$ during the time that the runner has no foot in contact with the ground.

$$\int_0^{0.1} (N(t)\mathbf{j} - T(t)\mathbf{i})dt + \int_0^{0.3} (-mg\mathbf{j} + F_D\mathbf{i})dt = \mathbf{0}$$

It follows that

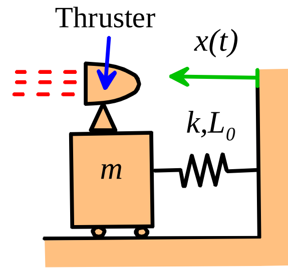
$$\int_0^{0.1} N(t)dt - 0.3mg = 0$$

The average force is

$$\frac{1}{0.1}\int_0^{0.1} N(t)dt = \frac{0.3}{0.1}mg$$

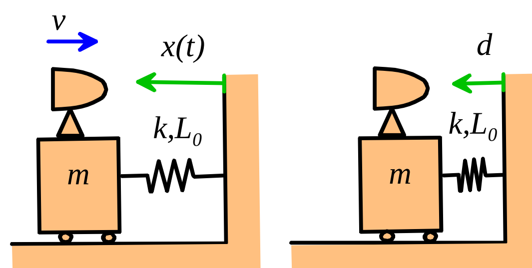
For a person with mass 80kg, the average vertical force comes out to be 7.8kN.

2. As background to this problem, read Koizumi, Komurasaki and Arakawa, (2004) ‘Development of thrust stand for low impulse measurement from micro-thrusters,’ *review of scientific instruments* **75** (10), 3185-3190. The goal of this problem is to illustrate the principle of their instrument using a simplified idealization, shown in the figure. Assume that



1. The system starts at rest, with $x = L_0$, where L_0 is the unstretched length of the spring.
2. At $t=0$ the thruster is fired. The thruster exerts a force on the mass for only a very short time, so the mass can be assumed to be stationary during firing.
3. The thruster starts the mass moving to the right. The length of the spring $x(t)$ is recorded, and in particular the length *d* at the instant of maximum spring compression is measured.

a. Use energy conservation to write down a relationship between the displacement of the mass at the instant of maximum compression of the spring to its speed just after the thruster fired.



We consider the thruster, mass and spring together as a conservative system. Equating the kinetic and potential energy just after the thruster is fired, and at the instant that the mass comes to rest, we have that

$$\frac{1}{2}mv^2 = \frac{1}{2}k(d - L_0)^2$$

Hence

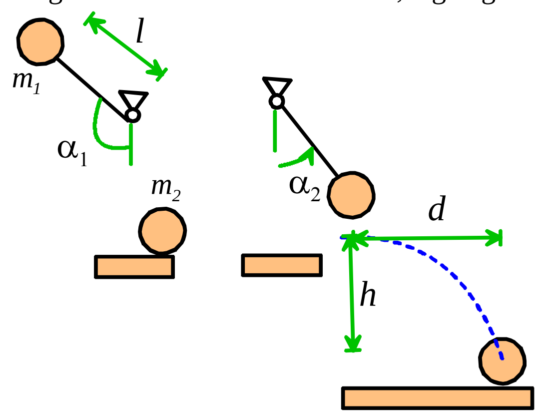
$$v = \sqrt{\frac{k}{m}}(L_0 - d)$$

b. Hence, calculate a formula for the impulse of the thruster, in terms of *d*, L_0, *k* and *m*.

The impulse can be calculated from the momentum of the mass just after the thruster is fired, as

$$I = mv = \sqrt{mk}(L_0 - d)$$

3. The figure shows an experimental apparatus for measuring the restitution coefficient of, e.g. a golf-ball or a bowling ball. It uses the following procedure



- A pendulum (a golf-club head, e.g.) is swung from a known initial angle α_0 and dropped from rest so as to strike the ball.
- The angle of follow-through α_1 of the pendulum is recorded
- The distance traveled by the ball d is measured

Your goal is to derive a formula that can be used to determine the restitution coefficient e from the measured data.

a. By considering its trajectory, derive an expression for the velocity of the ball just after it is struck, in terms of the height h, the measured distance d and the gravitational acceleration.

Let $\mathbf{v}_0 = v_b \mathbf{i}$ denote the velocity of the ball just after impact. The position vector as a function of time follows from the equations governing motion of a projectile under gravity

$$\mathbf{r} = v_b t\mathbf{i} - \frac{1}{2} g t^2 \mathbf{j}$$

At impact with the ground the position vector is $\mathbf{r} = d\mathbf{i} - h\mathbf{j}$ This gives two equations for the unknown time of impact and the speed v

$$v_b t = d \qquad h = \frac{1}{2} g t^2$$

These can be solved for d, with the result

$$v_b = \sqrt{\frac{gd^2}{2h}}$$

b. Derive an expression for the speed of the mass on the end of the pendulum just after it strikes the ball, in terms of α_1, l and the gravitational acceleration.

The pendulum can be idealized as a conservative system. Equating the kinetic and potential energies at the instant just after the collision and at the point when the pendulum comes to a stop at the top of its swing gives

$$\frac{1}{2} m_2 v_p^2 - m_2 g l = -m_2 g l \cos\alpha_2$$
$$\Rightarrow v_p = \sqrt{2gl\left(1 - \cos\alpha_2\right)}$$

c. Derive an expression for the speed of the mass on the end of the pendulum just before it strikes the ball, in terms of α_1, l and the gravitational acceleration.

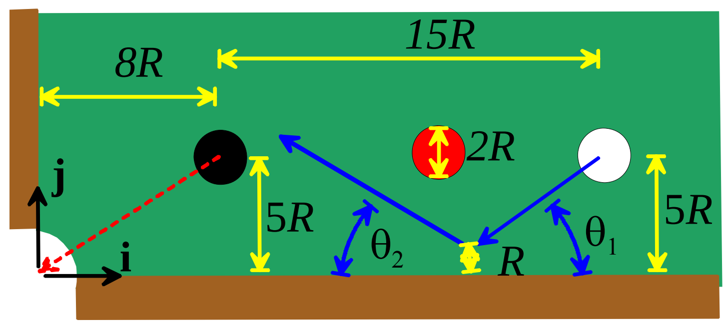
$$-m_2 g l \cos\alpha_2 = \frac{1}{2} m_2 v_0^2 - m_2 g l$$
$$\Rightarrow v_0 = \sqrt{2gl\left(1 - \cos\alpha_1\right)}$$

d. Hence, deduce a formula for the coefficient of restitution.

The restitution coefficient is

$$e = \frac{v_b - v_p}{v_0} = \frac{\sqrt{gd^2/2h} - \sqrt{2gl\left(1-\cos\alpha_2\right)}}{\sqrt{2gl\left(1-\cos\alpha_1\right)}}$$

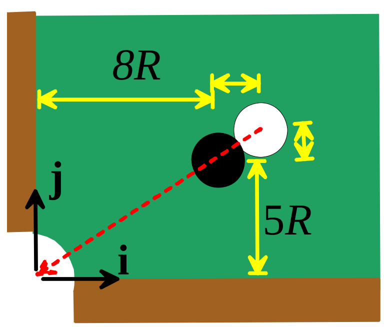
5. **Virtual Snooker**. The figure shows three balls on a snooker table. All balls have radius *R*. The impact of the cue ball (white) with the cushion has a restitution coefficient of 0.8; the impact of two balls has a restitution coefficient of 0.9. Your goal is to calculate an initial velocity for the (white) cue ball that will pot the black ball. Note that you are not allowed to hit the red ball, and so can't make a direct shot – you will have to play the shot off the cushion.



a. Write down the coordinates of the cue ball at the instant that it strikes the black ball, assuming that it is on a trajectory to knock the black ball directly into the pocket.

The direction of motion of the black ball after impact is parallel to the line connecting the centers of the two balls at impact, as shown in the figure. Straightforward geometry gives the position vector at impact as

$$\mathbf{r} = 8\left(1+\frac{2}{\sqrt{89}}\right)\mathbf{i} + 5\left(1+\frac{2}{\sqrt{89}}\right)\mathbf{j}$$



b. Consider the rebound of the cue ball off the cushion, as shown in the figure. Show that the angle of incidence is related to the angle of rebound by $\tan\theta_2 = e\tan\theta_1$.

Let $\mathbf{v} = v_{x0}\mathbf{i} + v_{y0}\mathbf{j}$ denote the velocity just before impact. Clearly $\tan\theta_1 = v_{y0}/v_{x0}$

After impact, the velocity components are $\mathbf{v} = v_{x1}\mathbf{i} + v_{y1}\mathbf{j} = v_{x0}\mathbf{i} - ev_{y0}\mathbf{j}$, and furthermore $\tan\theta_2 = -v_{y1}/v_{x1}$. Therefore

$$\tan\theta_2 = ev_{y0}/v_{x0} = e\tan\theta_1$$

as required.

c. Hence, calculate the initial direction of motion of the cue ball in order to make the shot.

Geometry shows that

$$R(15-16/\sqrt{89}) = \frac{4R}{\tan\theta_1} + \frac{R(4+10/\sqrt{89})}{\tan\theta_2} = \frac{4R}{\tan\theta_1} + \frac{R(4+10/\sqrt{89})}{e\tan\theta_1}$$

$$\Rightarrow \tan\theta_1 = \frac{(4+(4+2/\sqrt{89})/e)}{(15-16/\sqrt{89})} = 0.7761$$

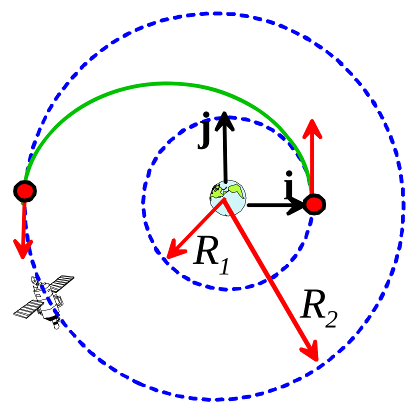
d. To check your calculation click here to download a MATLAB simulation of the shot. Save the code in a file called *snooker.m* and then run it with

```
>>snooker(Vx,Vy)
```

where Vx, Vy are the initial velocities of the cue ball (the magnitude is arbitrary – in fact the code will scale your velocity to have unit magnitude for convenience in doing the animation).

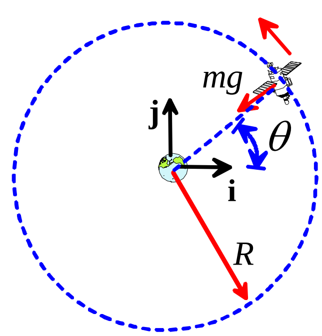
e. Set up a trick shot of your own design, and edit the MATLAB code to animate it. The code is a bit flakey – MATLAB will occasionally miss a collision, for example – so don't be too ambitious. If you email your MATLAB code to Stephanie_Gesualdi@brown.edu we will web post your shot…

6. A 'Hohmann Transfer' is a maneuver for changing the radius of a satellite orbit. It uses the following procedure:



- The satellite typically starts in a low circular orbit, with radius R_1
- At some convenient time, a rocket is fired to increase the speed of the satellite, without changing its direction of motion. This places the satellite on an elliptical orbit, with perigee R_1 and apogee R_2
- The satellite is allowed to complete one half of the elliptical orbit. When it reaches its apogee, a rocket is fired again, to increase its speed a second time, without changing its direction. This places the satellite in a circular orbit, with radius R_2

a. By drawing a free body diagram for the satellite, and using Newton's law of motion, calculate a formula relating the speed of a satellite v to the radius of its orbit, in terms of the gravitational parameter $\mu = GM$.



Following the usual procedure, the position, velocity and acceleration vector of the satellite are

$$\mathbf{r} = R(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

$$\mathbf{v} = R\frac{d\theta}{dt}(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = v(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})$$

$$\mathbf{a} = -R\left(\frac{d\theta}{dt}\right)^2(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) = -\frac{v^2}{R}(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

Newton's law gives

$$\mathbf{F} = m\mathbf{a} = -\frac{GMm}{R^2}(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) = -m\frac{v^2}{R}(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

This gives

$$R = \frac{\mu}{v^2}$$

b. Consider the satellite in its elliptical transfer orbit. Use energy conservation and angular momentum conservation to calculate a formula for the speed of the satellite v_p at the perigee of the satellite orbit, and the speed v_a at the apogee of the satellite orbit, in terms of $\mu = GM$, and R_1, R_2

Energy conservation relates v_a, v_p, $\mu = GM$, and R_1, R_2 as follows

$$\frac{1}{2}mv_a^2 - \frac{GMm}{R_a} = \frac{1}{2}mv_p^2 - \frac{GMm}{R_p}$$

Angular momentum conservation relates v_a, v_p and R_1, R_2 as follows

$$mv_aR_a = mv_pR_p$$

These two equations can be solved for v_a, v_p

```
> eq1 := va^2/2 - mu/Ra = vp^2/2-mu/Rp;
```

$$eq1 := \frac{va^2}{2} - \frac{\mu}{Ra} = \frac{vp^2}{2} - \frac{\mu}{Rp}$$

```
> eq2 := va*Ra = vp*Rp;
```

$$eq2 := va\ Ra = vp\ Rp$$

```
> solve({eq1,eq2},{va,vp});
```

$$\{vp = \mathrm{RootOf}((Ra\ Rp + Rp^2)\ \_Z^2 - 2\ \mu\ Ra),$$
$$va = \frac{\mathrm{RootOf}((Ra\ Rp + Rp^2)\ \_Z^2 - 2\ \mu\ Ra)\ Rp}{Ra}\}$$

```
> convert(%,radical);
```

$$\left\{vp = \sqrt{2}\sqrt{\frac{\mu\ Ra}{Ra\ Rp + Rp^2}},\ va = \frac{\sqrt{2}\sqrt{\frac{\mu\ Ra}{Ra\ Rp + Rp^2}}\ Rp}{Ra}\right\}$$

so that

$$v_p = \sqrt{\frac{2GMR_a}{R_p(R_p + R_a)}} \qquad v_a = \sqrt{\frac{2GMR_p}{R_a(R_p + R_a)}}$$

c. Hence, calculate formulae for the changes in speed of the satellite during the two rocket burns

It follows that

$$\Delta v_p = \sqrt{\frac{2GMR_a}{R_p(R_p + R_a)}} - \sqrt{\frac{GM}{R_p}} \qquad \Delta v_a = \sqrt{\frac{GM}{R_a}} - \sqrt{\frac{2GMR_p}{R_a(R_p + R_a)}}$$

d. A satellite in `Geo-synchronous' orbit around the earth completes one orbit in 24 hours. Calculate the radius of this orbit.

For this orbit, we know that $\frac{d\theta}{dt} = \frac{2\pi}{24 \times 60 \times 60}$ radians per second.

We also know that $v = R\frac{d\theta}{dt}$ and $R = \frac{GM}{v^2}$, which shows that

$$v^3 = GM\frac{d\theta}{dt} \Rightarrow v = \left(GM\frac{d\theta}{dt}\right)^{1/3}$$

With numbers $\mu = GM = 3.986012 \times 10^5$ km^{3}s^{-2} we find v=*3.07*km/s. The radius of the orbit follows as R=*42241* km.

e. Hence, calculate the changes in speed required to raise a satellite from low earth orbit (250km above the earth's surface) to geosynchronous orbit.

Taking the earth's radius as 6378.145km, we find

$$\Delta v_p = 2.44 km/\sec \qquad \Delta v_a = 1.47 km/\sec$$