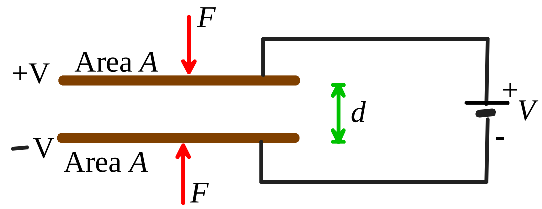
Division of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 4: Work and Energy Methods
MAX SCORE 55 POINTS

1. Two parallel plates in an electrostatic actuator experience an attractive force $F = AV^2\varepsilon/(2d^2)$ where ε is the permittivity of the medium between them.



a. Suppose that the upper plate of the actuator is displaced to decrease the separation from d_0 to d_1. Find a formula for the work done by the electrostatic force on the upper plate.

$$W = \int_{d_0}^{d_1} \mathbf{F}\cdot d\mathbf{r} = \int_{d_0}^{d_1} -\frac{AV^2\varepsilon}{2x^2}\mathbf{i}\cdot dx\mathbf{i} = \frac{AV^2\varepsilon}{2}\left(\frac{1}{d_1}-\frac{1}{d_0}\right)$$

[2 POINTS]

b. A typical actuator has $A = 10^{-7}m^2$, $d_0 = 10^{-7}m$ and is operated at 5V. Estimate the maximum work that can be done by closing the plates of the actuator. Assume that the gap between the plates is filled with air.

The permittivity of air (or free space) can be found on Google – it is $8.85\times10^{-12}\,\mathrm{m}^{-3}\ \mathrm{kg}^{-1}\ \mathrm{s}^4\ \mathrm{A}^2$

Substituting numbers into (a) gives $W = 0.11\times10^{-9}\,J$

[1 POINT]

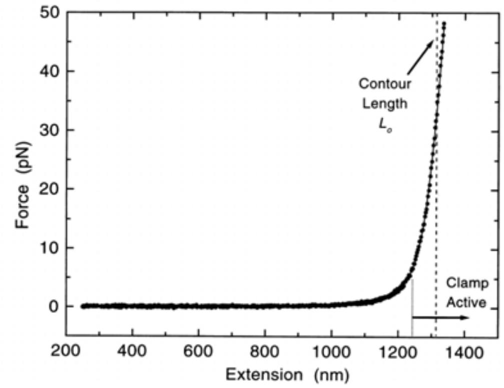
c. Suppose that the actuator operates at 1 kHz (1000 cycles per second) with $d_1 = 10^{-6}$m. Estimate (roughly) the mechanical power that could be generated by the device.

Every cycle, the actuator can do work $\dfrac{AV^2\varepsilon}{2}\left(\dfrac{1}{d_1}-\dfrac{1}{d_0}\right)$. The power is the work done per second, i.e. the frequency multiplied by the work done per cycle. Substituting numbers gives $P = 0.996\times10^{-7}W$

[2 POINTS]

[TOTAL 5 POINTS]

2. The figure shows the measured force-extension relation for a DNA molecule (the figure is from Wang *et al*, (1997), Biophysical Journal, **72**, 1335-1346). For low stretches, (less than about 1250nm), the forces exerted by the molecule are generated by its thermal vibration – much as pressure in a gas is generated by thermal motion of the atoms. In this regime the force-extension relation for the molecule can be approximated as



$$F = \left(\frac{kT}{L_p}\right)\left(\frac{1}{4(1-x/L_0)} - \frac{1}{4} + \frac{x}{L_0}\right)$$

where, k is the Boltzmann constant, T is absolute temperature (the product kT quantifies energy of thermal vibration in the system), L_p is the 'persistence length' of the molecule (the correlation length of the thermal vibrations), and L_0 is the total length of the molecule.

a. Find a formula for the work done to stretch the molecule from length $x=0$ to length a (use MAPLE to do the integral).

> `kT/Lp*(1/(4*(1-x/L))- ¼ + x/L); integrate(%,x=0..a);`

$$\frac{kT\left(\frac{1}{4 - \frac{4x}{L}} - \frac{1}{4} + \frac{x}{L}\right)}{Lp}$$

$$-\frac{1}{4}\,\frac{kT\left(\ln\left(\frac{4(L-a)}{L}\right)L^2 + a\,L - 2\,a^2 - 2\ln(2)\,L^2\right)}{Lp\,L}$$

> `combine(%,ln);`

$$-\frac{1}{4}\,\frac{kT\left(L^2\ln\left(\frac{L-a}{L}\right) + a\,L - 2\,a^2\right)}{Lp\,L}$$

This simplifies to

$$\int_0^a \mathbf{F}\cdot d\mathbf{x} = \left(\frac{kT}{L_p}\right)\int_0^a\left(\frac{1}{4(1-x/L_0)} - \frac{1}{4} + \frac{x}{L_0}\right)dx = \left(\frac{kT}{L_p}\right)\left[-\frac{L_0}{4}\log\left(1-a/L_0\right) - \frac{a^2}{2L_0} + \frac{a}{4}\right]$$

[3 POINTS]

b. Using data $L_p = 43$ nm, $L_0 = 1317$ nm, calculate the work required to stretch a molecule to half its total length, at room temperature (288 Kelvin).

Taking k=1.3806503 × 10^{-23} m^2 kg s^{-2} K^{-1} and substituting numbers gives $W = -0.211\times10^{-19}\,J$

[2 POINTS]

[TOTAL 5 POINTS]

3. The table below shows specifications for an electric motor (available for purchase from http://www.motortech.com/dcmotorCIR_MD.htm.)

Voltage	No-load Speed (rpm)	Stall torque (oz-in)	Stall current (A)	No load current (A)
27	16400	5.31	3.6	0.17

a. Estimate the maximum power that can be generated by the motor.

The maximum power can be calculated from the stall torque and the no-load speed as

$$P_{\max} = T_s \omega_{nl} / 2$$

Substituting numbers (**be sure to use radians per second and Nm)** gives 64.39 W

[2 POINTS]

b. Calculate values for the parameters R, β, T_0 and τ_0 in the equations describing the behavior of an electric motor.

From the notes, these expressions are given by

$$R = V / I_s \qquad \beta = \frac{V(I_s - I_{nl})}{I_s \omega_{nl}}$$

$$T_0 = \frac{V(I_s - I_{nl})}{\omega_{nl}} - T_s \qquad \tau_0 = \frac{T_s}{\omega_{nl}} - \frac{V(I_s - I_{nl})^2}{I_s \omega_{nl}^2}$$

Substituting numbers gives

$$\begin{array}{ll} R = 7.5 \ \text{Ohms} & \beta = 0.014979 \quad V/s \\ T_0 = 0.01643 Nm & \tau_0 = 0 Nms \end{array} \qquad (1.)$$

(the value of τ_0 comes out to $-.8084 \times 10^{-5}$ but τ_0 must be positive, so this is zero to the precision of the data provided).

[4 POINTS]

c. Determine the speed that maximizes the efficiency of the motor.

The efficiency is

$$\eta = \frac{T_s(1 - \omega / \omega_{nl})\omega}{VI} = \frac{T_s(1 - \omega / \omega_{nl}) R\omega}{V(V - \beta\omega)}$$

For the particular case where $\tau_0 = 0$, the speed that maximizes the efficiency can be calculated by differentiating η with respect to ω - it is easiest to do the calculation with MAPLE

```
> restart:
> Ts*(1-w/wnl)*R*w/(V*(V-beta*w));
```

$$\frac{Ts\left(1 - \frac{w}{wnl}\right) R\, w}{V\,(V - \beta\, w)}$$

```
> diff(%,w):
> solve(%=0,w);
```

$$\frac{2V + 2\sqrt{V^2 - \beta\, wnl\, V}}{2\beta}, \frac{2V - 2\sqrt{V^2 - \beta\, wnl\, V}}{2\beta}$$

The first solution has $\omega > \omega_{nl}$ and so the second one is the only solution of interest – substituting numbers gives $\omega = 1411$ rad/sec, or 13472 rpm. The efficiency is low – less than 50%. This is typical for a small brushed electric motor.

[**4 POINTS]**

[**TOTAL 10 POINTS]**

4. Estimate the power developed by the motor that lifts one of the elevators in the sciences library. Assume that you are the sole occupant. Explain briefly how you estimated values for relevant variables in the calculation.

The power developed is your weight*speed (assuming that the elevators own weight is balanced by counter-weights) Speed can be estimated by measuring the time to travel from bottom to top floor. It's also written inside the cabs – although the number quoted is suspect.

It turns out that the B&H elevators were recently renovated – including installation of new motors, and a regenerative energy recovery system. It won an award & there's a whole slide show on it at http://www.elevatorworld.com/Extras/01_09/extras.html, a full magazine article at http://www.elevator-world.com/files/sep08_copy.pdf
and some publicity from the company that did the work at http://www.elevatordrives.com/press/EV_2_2_2009.htm

Any sensible procedure to do the calculation should get credit.

[**5 POINTS]**

5. Bungee jumps are subject to extensive legal restrictions – see for example the Ohio standards at http://codes.ohio.gov/oac/901:9-1, and in particular http://codes.ohio.gov/oac/901%3A9-1-29.

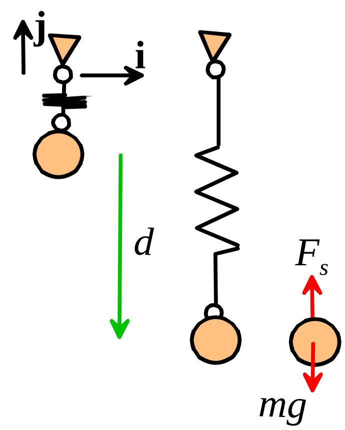
The goal of this problem is to establish a design procedure that will ensure that the standards are met. To provide some perspective, you will find it helpful to read Kockelman, J.W. and Hubbard, M. (2004) "Bungee jumping cord design using a simple model," Sports Engineering, **7**, 89.

Assume that the jumper has mass *m*, and the bungee cord has cross-sectional area *A*, unstretched length L_0, and is made from a material with Young's modulus *E* and tensile strength $\sigma_{\max}$. (If you have not done EN30, you may need to read Section 10 of the EN30 lecture notes at http://www.engin.brown.edu/courses/en3/notesframe.htm for definitions of *E* and $\sigma_{\max}$)

a. **The physics:** Using energy conservation, the equation of motion for the jumper at the instant that the cord is stretched to its maximum extent, and the force-extension equations for an elastic two force member, show that

- The maximum acceleration during the jump is $a_{\max} = g\sqrt{1+\frac{2AE}{mg}}$. Note that this means that a light person will experience higher accelerations than a heavy one, and also, surprisingly, the acceleration does not depend on the height of the jump or the length of the bungee cord.
- The jumper falls through a total distance $d = L_0\left(1+\frac{mg}{AE}+\frac{mg}{AE}\sqrt{1+\frac{2AE}{mg}}\right)$
- The maximum tensile stress (force/area) in the cord is $\frac{F}{A} = \left(\frac{mg}{A}+\frac{mg}{A}\sqrt{1+\frac{2AE}{mg}}\right)$

Force-extension relation A two force member with Young's modulus E, cross sectional area A and length L_0 behaves like a linear spring, with stiffness $k = AE/L_0$



Energy conservation. The system consists of the jumper and the cord together. Before the jump, and at the instant of maximum cable extension, the jumper is stationary and therefore has zero kinetic energy. The system is conservative and therefore the potential energy is constant. This shows that

$$\frac{1}{2}k(d-L_0)^2 - mgd = 0$$

Equation of motion. The figure shows a free body diagram for the jumper at the instant of maximum cord extension. Newton's law gives

$$\left(F_s - mg\right)\mathbf{j} = ma_{\max}\mathbf{j}$$

where $F_s = k(d-L_0)$ is the force exerted by the cord on the jumper.

The second and third equation can be solved for $a_{\max}$ and d – here's the MAPLE solution

```
> eq1 := k*(d-L0)^2/2-m*g*d=0: eq2 := k*(d-L0)-m*g=m*a:
> solve({eq1,eq2},{a,d});
```

$$\{d = \mathrm{RootOf}(k\,\_Z^2 + (-2\,k\,L0 - 2\,m\,g)\,\_Z + k\,L0^2),$$
$$a = -\frac{-k\,\mathrm{RootOf}(k\,\_Z^2 + (-2\,k\,L0 - 2\,m\,g)\,\_Z + k\,L0^2) + k\,L0 + m\,g}{m}\}$$

```
> convert(%,radical);
```

$$\{d = \frac{2\,k\,L0 + 2\,m\,g + 2\sqrt{2\,k\,L0\,m\,g + m^2\,g^2}}{2\,k},\ a = \frac{\sqrt{2\,k\,L0\,m\,g + m^2\,g^2}}{m}\}$$

```
> subs(k=A*E/L0,%);
```

$$\{d = \frac{L0\,(2\,A\,E + 2\,m\,g + 2\sqrt{2\,A\,E\,m\,g + m^2\,g^2}\,)}{2\,A\,E},\ a = \frac{\sqrt{2\,A\,E\,m\,g + m^2\,g^2}}{m}\}$$

These are the results stated. The force in the cable follows as $F_s = k(d-L_0) = AE(d-L_0)/L_0$, which again gives the required solution.

[5 POINTS]

b. **Design constraints:** The standards require that the maximum acceleration of the jumper must not exceed αg, where $\alpha = 3.5$ for a waist and chest harness, and $\alpha = 2.5$ for an ankle harness (the codes actually prescribe 'g forces' but this is what they really mean). They also require the maximum stress in the cable to be at least a factor of five lower than the tensile strength of the cable. Assume that the mass of the jumper will lie in the range $m_0 \le m \le m_1$. Show that the design constraints require that:

- For the acceleration to be within limits for all jumpers $\dfrac{gm_0}{AE} \ge \dfrac{2}{(\alpha^2 - 1)}$
- For the force to be within limits for all jumpers $\dfrac{gm_1}{AE} \le \left(\dfrac{\sigma_{max}}{5E}\right)^2 \dfrac{1}{2[1+\sigma_{max}/(5E)]}$
- For the jump to be feasible for any jumper, the material in the cord must have strength/stiffness ratio $\sigma_{max}/E > 10/(\alpha - 1)$

The lightest jumper will experience the largest acceleration. Therefore

$$\alpha g \ge g\sqrt{1 + \frac{2AE}{m_0 g}} \Rightarrow \frac{gm_0}{AE} \ge \frac{2}{(\alpha^2 - 1)}$$

[2 POINTS]

The heaviest jumper will induce the largest force in the cable. Consequently

$$\sigma_{max} \ge 5\left(\frac{m_1 g}{A} + \frac{m_1 g}{A}\sqrt{1 + \frac{2AE}{m_1 g}}\right)$$

$$\Rightarrow \frac{\sigma_{max}}{5E} \ge \frac{m_1 g}{AE} + \frac{m_1 g}{AE}\sqrt{1 + \frac{2AE}{m_1 g}}$$

The second equation can be solved for $\lambda = \dfrac{m_1 g}{AE}$ giving the result stated (here's the MAPLE method of solution but it's quite straightforward to solve by hand too)

```
> s = lambda + lambda*sqrt(1+2/lambda);
```

$$s = \lambda + \lambda\sqrt{1 + \frac{2}{\lambda}}$$

```
> solve(%,lambda);
```

$$\frac{s^2}{2\,(1+s)}$$

[3 POINTS]

The jump is feasible only if it is possible to meet both constraints for at least one weight of jumper. This gives

$$\frac{2}{(\alpha^2 - 1)} \le \left(\frac{\sigma_{max}}{5E}\right)^2 \frac{1}{2[1+\sigma_{max}/(5E)]}$$

This can be solved for $s = \sigma_{max}/5E$

```
> s^2/(2*(1+s))=2/(alpha^2-1);
```

$$\frac{s^2}{2+2\,s} = \frac{2}{\alpha^2 - 1}$$

```
> solve(%,s);
```

$$\frac{2}{\alpha-1}, -\frac{2}{\alpha+1}$$

This gives the required constraint.

[2 POINTS]

c. **Design Decision:** Suppose that the cord is to be made from rubber with E=0.7MPa and $\sigma_{\max} = 30MPa$. Choose suitable values for the maximum and minimum allowable weight for the jumper and recommend a value for the cross-sectional area of the cord that will ensure that the design limits are not exceeded (the solution is not unique – use your judgement).

d. A weight range of between 70lb and 250lb (31 kg to 113 kg) is reasonable – 70lb is the average weight of a 10 year old so restricting jumps to age 14 or older would be a good approach. People weighing more than 250lb should probably not be bungee jumping…

The design constraints can be expressed as a range of allowable values for A –

$$\frac{(\alpha^2-1)gm_0}{2E} \geq A \geq \frac{2\left[1+\sigma_{\max}/(5E)\right]gm_1}{E(\sigma_{\max}/5E)^2}$$

Substituting values gives

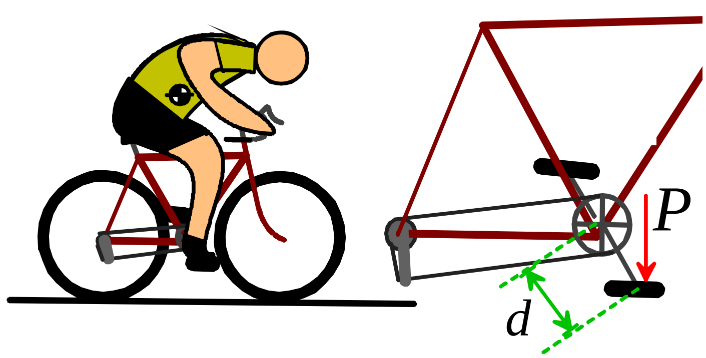
1. $0.00244 \geq A \geq 0.0004126$ (in m^2) for a waist/chest harness
2. $0.00114 \geq A \geq 0.0004126$ (in m^2) for an ankle harness

It is probably a good idea to choose a cord that will work for both types of harness; and to select a cross-sectional area that will err on the side of inducing a large acceleration rather than coming close to the failure load. A value of $A = 0.0009$ m^2 is reasonable (this is a cord with radius 1.7cm)

[3 POINTS]

TOTAL 15 POINTS

7. A bicycle travels at constant speed v. Assume that
(i) The rider exerts a vertical force P on the downward moving pedal;
(ii) The pedal cranks have length d,
(iii) The pedal cranks rotate at angular speed n revolutions per minute.
(iv) The bicyclist has frontal area A and drag coefficient C_D, and rides through air with density ρ.
(v) The air resistance of the bicycle can be neglected.

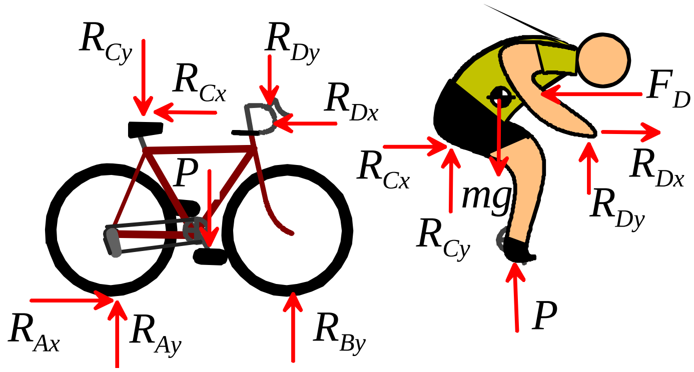


a. Draw a free body diagram showing all the forces acting on the bicycle (assume that the front wheel rolls freely)

FBD is shown in the figure. **[3 POINTS]**

b. Draw a free body diagram showing all the forces acting on the rider

FBD is shown in the figure **[3 POINTS]**



The two FBDs must be consistent – forces exerted on bike by rider must be equal and opposite to forces exerted on rider by bike. OK to neglect forces on handlebars.

c. Deduce that the horizontal force exerted by the rider on the bicycle is equal to $\rho C_D A v^2 / 2$

The rider moves at constant speed, and therefore the resultant force acting on him or her is zero.

$$(R_{Cx} + R_{Dx} - F_D)\mathbf{i} + (R_{Cy} + R_{Dy} + P - mg)\mathbf{j} = \mathbf{0}$$

Thus, $R_{Cx} + R_{Dx} = F_D$, where $F_D = \rho C_D A v^2 / 2$ is the magnitude of the air drag force.

[2 POINTS]

d. Calculate average rate of work done on the bicycle by the forces acting on the pedals, in terms of d and P

The force on each pedal moves through a vertical distance $2d$ during each revolution of the crank. The total work done is therefore $2dP$. There are two pedals, and they turn at n revolutions per minute – the power is therefore

$$W = 4Pdn / 60$$

[2 POINTS]

e. Calculate the rate of work done on the bicycle by all the other forces acting on the bicycle.

The reaction forces acting on the wheels are stationary and therefore do no work. The forces exerted by the rider on the bike move with horizontal speed $v\mathbf{i}$**,** and so do work

$$W = \left[-(R_{Cx} + R_{Dx})\mathbf{i} - (R_{Cy} + R_{Dy})\mathbf{j}\right] \cdot v\mathbf{i} = -(R_{Cx} + R_{Dx})v = -F_D v$$

[1 POINT]

f. Hence, show that the speed of the bicycle is related to the force exerted on the pedal and the rotational speed by

$$v = \left(\frac{2Pdn}{15\rho A C_D}\right)^{1/3}$$

The bike has no mass and has been idealized as a conservative system. The total rate of work done on the bike is zero, so

$$\frac{4Pdn}{60} - \frac{1}{2}\rho C_D A v^2 v = 0$$

This reduces to the expression given.

[2 POINTS]

g. Calculate the speed of the bicycle for A=1.5m^2, d=20cm, P=100N, $C_D = 0.5$, $\rho = 1.02 kgm^{-3}$, n=30

Substituting numbers gives 4.7 m/s, or 10.5 mph. This is not exactly yellow jersey speed but is consistent with a leisurely pace.

[1 POINT]

h. Calculate the required transmission ratio *v/n* for these operating conditions (the transmission ratio can be designed by selecting the bicycle's gears and the radius of the bike wheels appropriately – this will be discussed in more detail when we consider motion of rigid bodies.)

The transmission ratio is 4.7/30=0.157.

[1 POINT]

TOTAL 15 POINTS