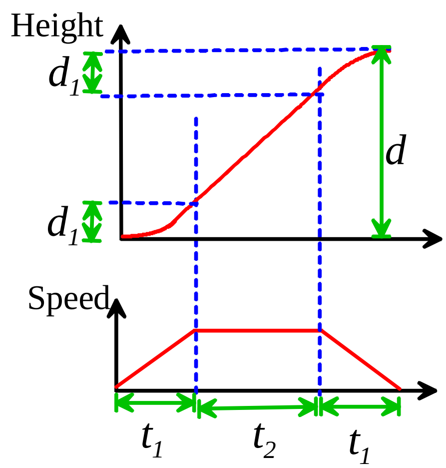
EN40: Dynamics and Vibrations

Homework 2: Dynamics of Particles
SOLUTIONS AND GRADING SCHEME. MAX SCORE 75

Division of Engineering
Brown University

1. The design specifications for an elevator specify (i) the maximum acceleration of the elevator $a_{\max}$; and (ii) the maximum speed of the elevator $V_{\max}$.

a. Derive a formula that predicts the shortest possible time for the elevator to travel between two floors that are a distance d apart, in terms of $a_{\max}$ $V_{\max}$.



To minimize the time of travel, the elevator must accelerate at its maximum rate until it reaches the maximum allowable speed; then travel at constant speed; then decelerate at the maximum allowable rate. The variation of the speed and height of the elevator as a function of time are sketched in the figure.

Note that

1. The position-velocity-acceleration relations are
$$\mathbf{r} = \left(x_0 + v_0(t-t_0) + \frac{1}{2}a(t-t_0)^2 \right)\mathbf{i}$$
$$\mathbf{v} = \left(v_0 + a(t-t_0) \right)\mathbf{i}$$
$$\mathbf{a} = a\mathbf{i}$$
2. The velocity-time relation shows that the time taken to accelerate to maximum speed (and also to decelerate to a standstill) is $t_1 = V_{\max}/a_{\max}$, and the distance traveled during this time is
$$d_1 = \frac{1}{2}a_{\max}t_1^2 = \frac{1}{2}\frac{V_{\max}^2}{a_{\max}} \qquad \textbf{(2 POINTS)}$$
3. If the total distance between floors is greater than $2d_1$, the time required to travel the remaining distance $d-2d_1$ at constant speed $V_{\max}$ is $t_2 = (d-2d_1)/V_{\max}$. The total time of travel is therefore
$$t = 2t_1 + t_2 = 2V_{\max}/a_{\max} + (d-2d_1)/V_{\max} = V_{\max}/a_{\max} + d/V_{\max} \qquad \textbf{(3 POINTS)}$$
4. If the total distance between floors is less than d_1, then the elevator will never reach its maximum speed. Under these conditions the elevator will travel half the distance between the two floors at maximum acceleration, and the remainder at maximum deceleration. The time to travel a distance *d/2* at maximum acceleration follows from the position-time relation as $t = \sqrt{d/a_{\max}}$. The total time to travel between floors is therefore $t = 2\sqrt{d/a_{\max}}$

The solution is therefore
$$t = \begin{cases} V_{\max}/a_{\max} + d/V_{\max} & d > V_{\max}^2/a_{\max} \\ 2\sqrt{d/a_{\max}} & d < V_{\max}^2/a_{\max} \end{cases}$$
(2 POINTS for the first solution, plus 2 points extra credit for anyone who gets both cases)

b. Using the data in the article http://www.elevator-world.com/magazine/archive01/0309-003.shtml, (and any other sources that you find helpful) estimate the maximum acceleration of the elevators in the Taipei tower.

The article gives the following data:

- The elevators in the Taipei tower travel at $V_{\max} = 1010$ mpm.
- The Taipei tower is 388 m high
- The elevator takes 38 seconds to travel from the bottom to the top floor.

The acceleration can be calculated from the time as

$$a_{\max} = \frac{V_{\max}}{t - d / V_{\max}}$$ **(2 POINTS)**

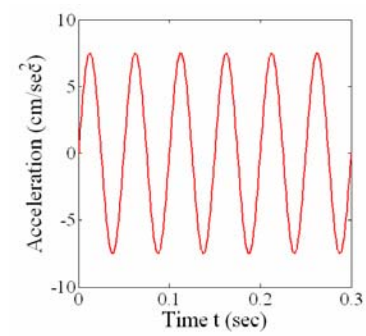
Substituting numbers gives $a_{\max} = 1.12 m/s^2$. **(1 POINT)**

The best-selling former Oprah book of the month "Elevator Traffic Handbook" by Gina Barney, page 84 http://books.google.com/books?id=GteIiGQT1S4C&printsec=frontcover&dq=elevator+traffic+handbook recommends an acceleration of $g/8 = 1.2 m/s^2$. This book also suggests limits on the "jerk" – this is the time derivative of the acceleration. It is likely that the design follows this handbook very closely.

TOTAL 10 POINTS + 2 EXTRA CREDIT

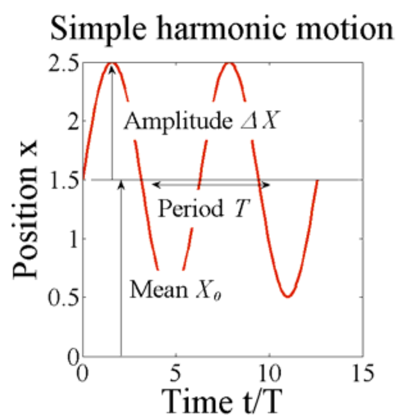
2. The figure shows a vibration measurement from an accelerometer. The vibration can be assumed to be harmonic. Determine

1. The amplitude of the acceleration
2. The period of oscillation
3. The angular frequency of oscillation (in radians per second)
4. The amplitude of the velocity of the accelerometer
5. The amplitude of the displacement of the accelerometer.



These terms are defined as shown in the figure below

From the figure, we estimate



- The amplitude of the acceleration is $7.5 cm/s^2$
- We see 6 complete cycles in 0.3 secs, so the period is 0.05sec
- The angular frequency follows as $\omega = 2\pi / T = 125 \, rad/s$
- The amplitude of the velocity is related to the amplitude of the acceleration by

$$\Delta A = \frac{2\pi}{T} \Delta V$$

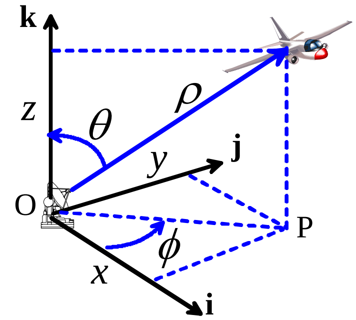
  It follows that $\Delta V = 0.0579 cm/s$
- The amplitude of the displacement is related to the amplitude of the velocity and acceleration by

$$\Delta V = \frac{2\pi}{T} \Delta X \qquad \Delta A = \left(\frac{2\pi}{T}\right)^2 \Delta X = \frac{2\pi}{T} \Delta V$$

  It follows that $\Delta X = 4.75 \times 10^{-5} cm = 0.0475 mm$

(1 POINT FOR EACH SOLUTION – TOTAL 5 POINTS)

3. A ground-based radar system is used to track the position of an aircraft. The radar records the distance $\rho(t)$ of the aircraft from the tracking station, and the two angles $\theta(t)$ and $\phi(t)$, as functions of time.



a. Using elementary trigonometry, write down the position vector of the aircraft as components in the basis shown, in terms of $\rho(t)$ and the two angles $\theta(t)$ and $\phi(t)$. To find the x and y components of position, it is helpful to first calculate the length of OP.

The length of OP is $\rho(t)\sin\theta(t)$. The position vector is therefore

$\mathbf{r} = \rho(t)\sin\theta(t)\cos\phi(t)\mathbf{i} + \rho(t)\sin\theta(t)\sin\phi(t)\mathbf{j} + \rho(t)\cos\theta(t)\mathbf{k}$ **(2 POINTS)**

b. Using MAPLE, calculate a formula for the velocity vector of the aircraft, in terms of $d(t)$ and the two angles $\theta(t)$ and $\phi(t)$, and their time derivatives

$$\frac{d\rho}{dt}, \frac{d\theta}{dt}, \frac{d\phi}{dt}$$

c. Use MAPLE to find a formula for the magnitude of the velocity vector.

Here's a MAPLE solution to b and c

```
> restart:
> with(VectorCalculus):
Warning, the assigned names <,> and <|> now have a global binding
Warning, these protected names have been redefined and unprotected: *,
+, ., Vector, diff, int, limit, series
> r :=
<rho(t)*sin(theta(t))*cos(phi(t)),rho(t)*sin(theta(t))*sin(phi(t)),rho
(t)*cos(theta(t))>;
```

$$r := \rho(t)\sin(\theta(t))\cos(\phi(t))\,\mathbf{e}_{\mathbf{x}} + \rho(t)\sin(\theta(t))\sin(\phi(t))\,\mathbf{e}_{\mathbf{y}} + \rho(t)\cos(\theta(t))\,\mathbf{e}_{\mathbf{z}}$$

```
> v := diff(%,t);
```

$$v := \left(\left(\frac{d}{dt}\rho(t)\right)\sin(\theta(t))\cos(\phi(t)) + \rho(t)\cos(\theta(t))\left(\frac{d}{dt}\theta(t)\right)\cos(\phi(t)) - \rho(t)\sin(\theta(t))\sin(\phi(t))\left(\frac{d}{dt}\phi(t)\right)\right)\mathbf{e}_{\mathbf{x}} + \left(\left(\frac{d}{dt}\rho(t)\right)\sin(\theta(t))\sin(\phi(t)) + \rho(t)\cos(\theta(t))\left(\frac{d}{dt}\theta(t)\right)\sin(\phi(t)) + \rho(t)\sin(\theta(t))\cos(\phi(t))\left(\frac{d}{dt}\phi(t)\right)\right)\mathbf{e}_{\mathbf{y}} + \left(\left(\frac{d}{dt}\rho(t)\right)\cos(\theta(t)) - \rho(t)\sin(\theta(t))\left(\frac{d}{dt}\theta(t)\right)\right)\mathbf{e}_{\mathbf{z}}$$

```
> simplify(sqrt(DotProduct(v,v)));
```

$$\sqrt{\left(\frac{d}{dt}\phi(t)\right)^2\rho(t)^2 - \left(\frac{d}{dt}\phi(t)\right)^2\rho(t)^2\cos(\theta(t))^2 + \left(\frac{d}{dt}\rho(t)\right)^2 + \left(\frac{d}{dt}\theta(t)\right)^2\rho(t)^2}$$

This expression can be simplified further to give

$$|\mathbf{v}| = \sqrt{\left(\rho\frac{d\phi}{dt}\sin\theta\right)^2 + \left(\frac{d\rho}{dt}\right)^2 + \left(\rho\frac{d\theta}{dt}\right)^2}$$ **(3 POINTS FOR b and c)**

(5 POINTS TOTAL)

4. Repeat the calculation done in class to estimate the minimum thrust that must be produced by the engines of an aircraft in order to take off from the deck of an aircraft carrier – but extend your calculations to account for the motion of the aircraft carrier.

a. Derive formulas for the required acceleration of the aircraft, and the time required to reach takeoff speed, in terms of the runway length d, the take-off speed v_t and the speed of the aircraft carrier V.

We are given (i) the length of the runway d, (ii) the takeoff speed v_t (iii) the aircraft is traveling at speed V at the start of the takeoff roll; (iv) the aircraft carrier travels at speed V. We can therefore write down the position vector **r** of the end of the aircraft carrier runway and the aircraft, as well as the aircraft velocity. Taking the origin at the initial position of the aircraft, we have that, at the instant of takeoff

$$\mathbf{r} = (d + Vt)\mathbf{i} = \left(Vt + \frac{1}{2}at^2\right)\mathbf{i} \qquad \mathbf{v} = v_t\mathbf{i} = (V + at)\mathbf{i}$$

Here, the first expression for **r** is the position vector of the end of the runway at the instant that the aircraft takes off – the Vt term accounts for the fact that the end of the runway is moving. The second expression is the position vector of the aircraft. Of course, the two are equal at takeoff.

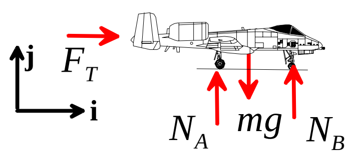
This gives two scalar equations which can be solved for a and t

$$d + Vt = Vt + \frac{1}{2}at^2 \quad v_t = V + at \quad \Rightarrow \quad a = \frac{(V - v_t)^2}{2d} \quad t = \frac{2d}{v_t - V}$$ **(5 POINTS)**

b. Draw a free body diagram and write down the equation of motion for the aircraft, expressing your solution in vector form.

EOM: The vector equation of motion for this problem is

$$F_T\mathbf{i} = ma\mathbf{i} = m\frac{(v_t - V)^2}{2d}\mathbf{i}$$ **(2 POINTS)**



c. Hence, calculate a formula for the required engine thrust.

The **i** component of the equation of motion gives an equation for the unknown force in terms of known quantities

$$F_T = m\frac{(v_t - V)^2}{2d}$$ **(1 POINT)**

d. Using data given in class, and taking V=35 knots, estimate a value for the required thrust. What is the percentage reduction in thrust due to the motion of the aircraft carrier?

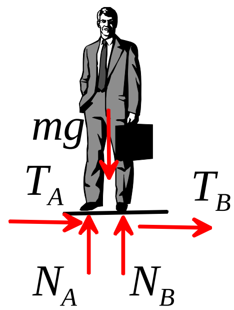
Substituting numbers gives 130kN – a much more reasonable safety factor than the value calculated in class! The motion of the aircraft carrier reduces the thrust by 58%. **(2 POINTS)**
Note that there are other ways to think about this problem – for example, you could use a reference frame that travels with the aircraft carrier. Newton's laws can be applied without modification in a translating reference frame. This shows immediately that the takeoff speed is simply reduced by the aircraft carrier speed.

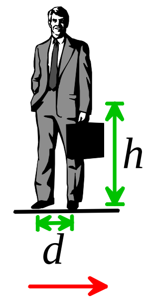
(10 POINTS)

5. Estimate the maximum allowable acceleration of a `people mover' to ensure that a person can remain standing as the vehicle starts and stops. (the figure is from www.leitner-lifts.com)

a. Draw a free body diagram showing the forces acting on a person in the vehicle. Assume that the person stands sideways, and (stupidly) has both hands free.

The figure is shown below





(3 POINTS)

b. Write down the equations of motion for the person (you will need both **F**=m**a** and **M**=**0. Be sure to take moments about the center of mass!!**). Express your answers in vector form, in terms of the height h of the person's center of mass above the ground, and the distance d between his or her feet.

The equations of motion are

$$\mathbf{F} = m\mathbf{a} \Rightarrow (T_A + T_B)\mathbf{i} + (N_A + N_B - mg)\mathbf{j} = ma\mathbf{i}$$
$$\mathbf{M}_C = \left[(T_A + T_B)h + (N_B - N_A)d/2 \right]\mathbf{k} = \mathbf{0}$$

(2 POINTS)

c. Hence, find a formula for the maximum acceleration in terms of h and d.

The equations of motion can be solved for the reaction forces (use MAPLE if you like) with the result

$$N_B = \frac{mg}{2} - \frac{mah}{d} \qquad N_A = \frac{mg}{2} + \frac{mah}{d}$$

The person is just on the point of tipping over when the reaction force at B is zero. The critical acceleration is therefore

$$a = \frac{dg}{2h}$$

(3 POINTS)

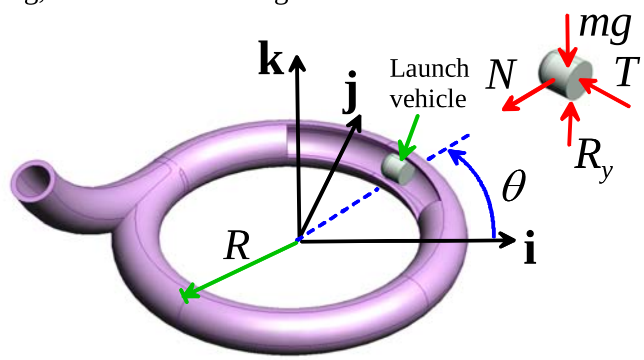
d. Estimate values for h and d and so calculate a value for the acceleration.

You could measure yourself – d is about 1.5 feet and h is about 3 feet for me. This gives $a = g/4$. You'd probably want to keep the acceleration at about half this value – so we get the same magic $a = g/8$ that is used for vertical elevators.

(2 POINTS)

TOTAL: 10 POINTS

6. There is currently great interest in the aerospace industry in developing alternatives to chemically powered rocket motors for launching satellites into space. One such proposal is to build a large circular electromagnetic rail gun. The gun would consist of a large vacuum tube (the vacuum is needed to eliminate air resistance), arranged in a circular ring. A magnetic levitation system would be used to keep the launch vehicle from touching the sides of the ring, and an electromagnetic linear motor would accelerate the satellite to launch speed. On reaching launch speed the vehicle is permitted to enter the exit tube – thereafter it flies as a ballistic missile. A very brief description of this facility, due to be tested in 2009, can be found at http://www.launchpnt.com/portfolio/space-launch.html



The goal of this problem is to work through some preliminary design calculations for the device: in particular, we will estimate the time required to accelerate the satellite to its launch speed, and the forces acting on the satellite as it travels around the ring.

Note that

- The position of the projectile can be conveniently described in terms of the angle $\theta(t)$ shown in the figure.
- The linear motor applies a tangential force T to the projectile, and a magnetic levitation system applies a normal force N that will force the projectile to travel in a circle. The tangential force T is constant, but N must vary with time.

a. Write down the position vector of the projectile in terms of the ring radius R and the angle $\theta(t)$.

$$\mathbf{r} = R\cos\theta\mathbf{i} + R\sin\theta\mathbf{j} \qquad \textbf{(1 POINT)}$$

b. Find a formula for the velocity vector of the projectile in terms of R and the time derivatives of $\theta(t)$.

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -R\frac{d\theta}{dt}\sin\theta\mathbf{i} + R\frac{d\theta}{dt}\cos\theta\mathbf{j} = R\frac{d\theta}{dt}(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) \qquad \textbf{(1 POINT)}$$

c. Find a formula for the magnitude of the velocity vector, in terms of R and the time derivatives of $\theta(t)$.

Noting that $(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})$ is a unit vector we see immediately that $|\mathbf{v}| = R\frac{d\theta}{dt}$ **(1 POINT)**

d. Find a formula for the acceleration vector of the projectile, in terms of R and the time derivatives of $\theta(t)$.

Differentiating the velocity vector shows that

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = R\frac{d^2\theta}{dt^2}(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) + R\frac{d\theta}{dt}(-\cos\theta\frac{d\theta}{dt}\mathbf{i} - \sin\theta\frac{d\theta}{dt}\mathbf{j})$$

$$= -R\left(\frac{d^2\theta}{dt^2}\sin\theta + \left(\frac{d\theta}{dt}\right)^2\cos\theta\right)\mathbf{i} + R\left(\frac{d^2\theta}{dt^2}\cos\theta - \left(\frac{d\theta}{dt}\right)^2\sin\theta\right)\mathbf{j}$$

(2 POINTS)

(You can do this with MAPLE too)

e. Write down the force vector acting on the projectile in terms of N and T, expressing your answer as components in the basis shown

Resolving the forces into components along **i** and **j** gives

$$\mathbf{F} = -N(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) + T(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) + (R_y - mg)\mathbf{k}$$
$$= -(N\cos\theta + T\sin\theta)\mathbf{i} + (-N\sin\theta + T\cos\theta)\mathbf{j} + (R_y - mg)\mathbf{k}$$

(2 POINTS)

f. Write down **F** =m **a** for the projectile, and combine the components of the equation of motion to show that

$$\frac{d^2\theta}{dt^2} = \frac{T}{mR} \qquad\qquad N = mR\left(\frac{d\theta}{dt}\right)^2$$

The equation of motion is

$$-(N\cos\theta + T\sin\theta)\mathbf{i} + (-N\sin\theta + T\cos\theta)\mathbf{j} + (R_y - mg)\mathbf{k}$$
$$= -Rm\left(\frac{d^2\theta}{dt^2}\sin\theta + \left(\frac{d\theta}{dt}\right)^2\cos\theta\right)\mathbf{i} + Rm\left(\frac{d^2\theta}{dt^2}\cos\theta - \left(\frac{d\theta}{dt}\right)^2\sin\theta\right)\mathbf{j}$$

(1 POINT FOR EOM)

The **i** and **j** components of the equations of motion give

$$-(N\cos\theta + T\sin\theta) = -Rm\left(\frac{d^2\theta}{dt^2}\sin\theta + \left(\frac{d\theta}{dt}\right)^2\cos\theta\right)$$

$$(-N\sin\theta + T\cos\theta) = Rm\left(\frac{d^2\theta}{dt^2}\cos\theta - \left(\frac{d\theta}{dt}\right)^2\sin\theta\right)$$

You can solve these for T and N using MAPLE, but it's easier just to multiply the first equation by $\cos\theta$ and the second by $\sin\theta$ and add them, which gives

$$-N(\cos^2\theta + \sin^2\theta) = -Rm\left(\frac{d\theta}{dt}\right)^2\left(\sin^2\theta + \cos^2\theta\right)$$

which (remembering that $\sin^2\theta + \cos^2\theta = 1$ gives the second equation of motion, and multiply the first equation by $\sin\theta$ and the second by $\cos\theta$ and subtract the second from the first, which gives

$$-T(\sin^2\theta + \cos^2\theta) = -Rm\frac{d^2\theta}{dt^2}\left(\sin^2\theta + \cos^2\theta\right)$$ **(2 POINTS FOR SOLUTION)**

which gives the second equation of motion.

g. Assume that the vehicle starts at rest with $\theta = 0$. Find expressions for $\theta(t)$ and $d\theta/dt$ in terms of time and the variables T, m and R. (You should be able to solve the equation of motion by hand, but if you are stuck, use the MAPLE 'dsolve' function). Hence deduce a formula for the variation of the normal force with time.

Here's how to do the integrals with separation of variables. This may or may not be the method you are used to – there are many other approaches.

$$\frac{d^2\theta}{dt^2} = \frac{T}{mR} \quad \Rightarrow \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{T}{mR}$$

$$\Rightarrow \int_0^{d\theta/dt} d\omega = \int_0^t \frac{T}{mR} dt \Rightarrow \frac{d\theta}{dt} = \frac{T}{mR} t$$

$$\Rightarrow \int_0^{\theta} d\theta = \int_0^t \frac{T}{mR} t dt \Rightarrow \theta = \frac{T}{2mR} t^2$$

For the faint of heart, here's the MAPLE solution

```
> restart:
> eom := {diff(theta(t),t$2) = T/(m*R)}:
> ICs := {theta(0)=0,D(theta)(0) = 0}:
> dsolve(eom union ICs, theta(t));
```

$$\theta(t) = \frac{T\,t^2}{2\,m\,R}$$

(2 POINTS FOR EITHER MAPLE OR HAND CALCULATION)

h. Using your solutions to (c) and (g), find a formula for the speed of the launch vehicle as a function of time.

The speed is simply the magnitude of the velocity, $v = R\frac{d\theta}{dt} = \frac{T}{m}t$ **(1 POINT)**

i. Using the following data, calculate (i) The total time required to accelerate the launch vehicle to launch speed; (ii) the total number of times the vehicle must travel around the ring; (iii) The magnitude of the maximum acceleration of the vehicle; (iv) The maximum value of the reaction force *N*.

- Ring radius *R* 1.5 miles
- Vehicle mass *m* 220 lb
- Linear motor thrust *T:* 300N
- Launch speed 2km/sec

The time taken to reach launch speed is $t = mv/T$. Substituting numbers gives 666s, or 11 minutes.

The value of $\theta(t)$ at this time follows from $\theta = Tt^2/(2mR)$. Substituting numbers gives $\theta = 276$. The number of revolutions is $276/2\pi = 44$.

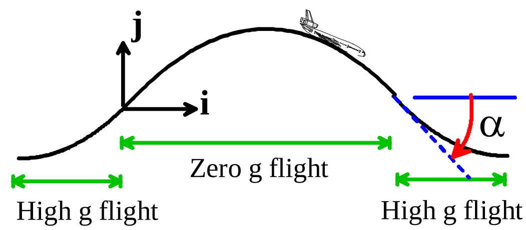
The maximum reaction force follows as $N = mR\left(\frac{d\theta}{dt}\right)^2$. Substituting numbers gives 166kN.

(2 POINTS)

TOTAL: 15 POINTS

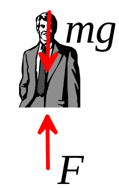
7. The `Vomit Comet' is a specially equipped aircraft used by NASA and others to simulate weightlessness for short periods of time (the aircraft is actually no longer a Comet – NASA retired their Comet in 2004. They now sub-contract flights to a commercial company http://www.gozerog.com/ who fly a Boeing 727). The goal of this problem is to estimate the maximum length of time that the weightless period can last; and to estimate the air-space required for the maneuver. Before attempting this problem, you should download and read the paper: F. Karmali and M. Shelhamer, (2008) "The dynamics of parabolic flight: Flight characteristics and passenger percepts," Acta Aeronautica, **63**, pp.594-602 (you can access this paper in Brown's database of electronic journals).

The figure shows the flight profile: it consists of a 'high-g' phase, followed by a 'zero-g' phase. During the 'zero g' phase the aircraft has constant horizontal speed, and has acceleration vector $\mathbf{a} = -g\mathbf{j}$.



a. Explain *briefly* (no more than 4 sentences, possibly with the aid of a free body diagram!) why the passengers in the aircraft feel weightless during the `zero-g' phase.

The body doesn't sense gravity directly – instead, it senses the deformation induced by internal forces in the body. The figure shows a free body diagram for a cross-section through a human body. When the body is at rest, and subjected to the earth's gravitational field, the internal forces must balance gravity – so $F=mg$ where m is the mass of the portion of the body above the cross-section. When the vertical acceleration of the body is –g, the internal forces are zero, and so the body does not sense the effects of gravity.



(3 POINTS FOR ANYTHING CONVINCING…)

b. Consider the flight of the aircraft during the zero-g phase. Assume that the aircraft enters the 'zero g' trajectory at time $t=0$ and at this instant has position vector $\mathbf{r} = \mathbf{0}$ and velocity vector $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j}$. Write down formulas for the velocity and position vectors as functions of time.

From the relations given in class:

$$\mathbf{r} = v_x t\mathbf{i} + \left(v_y t - \frac{1}{2}gt^2\right)\mathbf{j}$$
$$\mathbf{v} = v_x\mathbf{i} + \left(v_y - gt\right)\mathbf{j}$$
(1 POINT)

c. Using the following information, estimate (i) the total time of the zero-g portion of flight, (ii) the altitude change during zero-g flight; and (iii) the horizontal distance traveled by the aircraft during zero g flight.
   - The maximum allowable true airspeed speed during zero g flight is 360 knots
   - The minimum allowable true airspeed is 245 knots.

   1. The airspeed is the magnitude of the velocity, i.e. $|\mathbf{v}| = \sqrt{v_x^2 + \left(v_y - gt\right)^2}$

2. Clearly, the minimum speed occurs when $v_y = gt$, and the speed at this instant is v_x. We conclude that $v_x = 245$ knots.
3. The aircraft has maximum speed at the start and end of the parabolic zero-g portion of the flight. At time $t=0$ we have that $|\mathbf{v}| = \sqrt{v_x^2 + v_y{}^2} = 360$ knots. We can solve for v_y, which gives $v_y = 264$ knots.
4. Finally, we can calculate the time required for the vertical component of velocity to decrease to zero – this is $t = v_y / g$. Be careful with units – 264 knots is 136 m/s. We get $t=13.9$ sec. This is half the total time of zero g flight – so we get 27.8 sec of zero g flight. In practice it takes a few seconds to transition from high-g to zero-g so the actual zero g flight is slightly shorter than this.

(4 POINTS)

d. Calculate the pitch angle α of the aircraft at the end of the zero g trajectory

The pitch angle is $\tan^{-1}(v_y / v_x) = 47^0$ (**1 POINT)**

e. Is an engine thrust required to maintain "weightless" flight? If so, why?

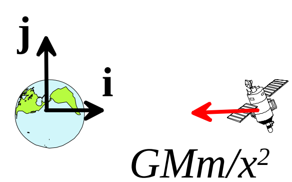
Yes – even though the aircraft is essentially in `free-fall' a small thrust is required to counteract the effects of drag. (**1 POINT)**

(TOTAL 10 POINTS)

8. The goal of this problem is to estimate (very crudely) the velocity required for the ballistic launch vehicle described in Problem 7 to reach Low Earth Orbit (LEO). Assume that

- The vehicle is launched in a direction perpendicular to the earth's surface (parallel to the **i** direction)
- Air resistance can be neglected
- The earth's rotation can be neglected.

a. Draw a free body diagram showing the forces acting on the launch vehicle.



A free body diagram is shown. **(1 POINT)**

b. Write down the force vector acting on the launch vehicle, expressing your answer in terms of the gravitational G, the mass of the earth M, the mass of the launch vehicle m, and the distance x.

The force vector is

$$\mathbf{F} = -\frac{GMm}{x^2}\mathbf{i}$$ **(1 POINT)**

c. Write down Newton's law of motion for the launch vehicle, and hence show that the distance from the earth satisfies the differential equation

$$\frac{d^2x}{dt^2} + \frac{GM}{x^2} = 0$$

The equation of motion is

$$\mathbf{F} = m\mathbf{a} \Rightarrow -\frac{GMm}{x^2}\mathbf{i} = m\frac{d^2x}{dt^2}\mathbf{i} \quad \textbf{(1 POINT)}$$

and the **i** component of this equation gives the required answer.

d. Show that the equation of motion can be re-written as

$$v_x \frac{dv_x}{dx} + \frac{GM}{x^2} = 0 \quad \text{where } v_x = \frac{dx}{dt}$$

The acceleration can be re-written as

$$\frac{d^2x}{dt^2} = \frac{dv_x}{dt} = \frac{dv_x}{dx}\frac{dx}{dt} = v_x\frac{dv_x}{dx}$$

Substituting this result into the equation of motion and rearranging it gives the result stated.

(3 POINTS)

e. Solve this equation of motion with initial conditions $v_x = V_0$ at x=R to obtain a formula for the speed of the launch vehicle v_x . You can do this by hand, or you can use the MAPLE 'dsolve' function.

The equation of motion can be integrated as follows

$$v_x \frac{dv_x}{dt} = -\frac{GM}{x^2} \Rightarrow \int_{V0}^{v_x} v_x dv_x = \int_R^x -\frac{GM}{x^2}dx \Rightarrow \frac{1}{2}\left(v_x^2 - V_0^2\right) = GM\left(\frac{1}{x} - \frac{1}{R}\right)$$

Hence

$$v_x = \sqrt{2GM\left(\frac{1}{x} - \frac{1}{R}\right) + V_0^2} \quad \textbf{(2 POINTS)}$$

f. Finally, estimate a value for the launch speed, assuming that

- The earth's radius is 6378.145km
- The Gravitational parameter $\mu = GM = 3.986012 \times 10^5$ km^{3}s^{-1}
- The LEO altitude is 250km above the earth's surface
- The vehicle has zero velocity when it reaches LEO altitude (a rocket must be fired at this instant to start the satellite in its orbit)

In practice a spacecraft would not be launched this way – instead it would be put on a path *tangent* to the earth's surface, placing it in an elliptical transfer orbit. You would still have to fire a rocket when the spacecraft reaches its maximum altitude (otherwise it will just return to the earth, which would scare the pants off the people in the launch facility). But a much smaller and lighter rocket motor is required with this method of launch.

Setting $v_x = 0$ allows us to solve for the launch velocity

$$V_0 = \sqrt{2GM\left(\frac{1}{x} - \frac{1}{R}\right)}$$

Substituting numbers gives 2.17 km/sec **(2 POINTS)**

TOTAL: 10 POINTS