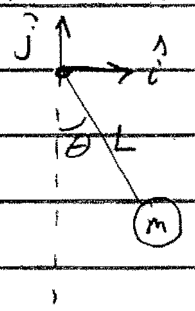
EN 4 HW 5 SOLUTIONS

1. The Pendulum. The position vector is



$$\underset{\sim}{r} = L\sin\theta\,\hat{i} - L\cos\theta\,\hat{j}$$

The velocity vector is then

$$\underset{\sim}{V} = L\frac{d\theta}{dt}\cos\theta\,\hat{i} + L\frac{d\theta}{dt}\sin\theta\,\hat{j}$$

The acceleration vector is then

(+2) $$\underset{\sim}{a} = \frac{d\underset{\sim}{V}}{dt} = L\left(\frac{d^2\theta}{dt^2}\cos\theta - \left(\frac{d\theta}{dt}\right)^2\sin\theta\right)\hat{i} + L\left(\frac{d^2\theta}{dt^2}\sin\theta + \left(\frac{d\theta}{dt}\right)^2\cos\theta\right)\hat{j}$$

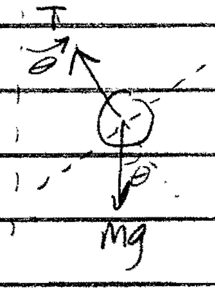
$\underset{\sim}{a}$ has two parts, a piece $\parallel$ to $\underset{\sim}{r}$ and a piece $\perp$ to $\underset{\sim}{r}$:
Rewriting,

$$\underset{\sim}{a} = -L\left(\frac{d\theta}{dt}\right)^2(\sin\theta\,\hat{i} - \cos\theta\,\hat{j}) \qquad \parallel \text{ to } \underset{\sim}{r}$$

(+1) $$+ L\frac{d^2\theta}{dt^2}(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) \qquad \perp \text{ to } \underset{\sim}{r}$$

Drawing the F.B.D., we have

(+2)

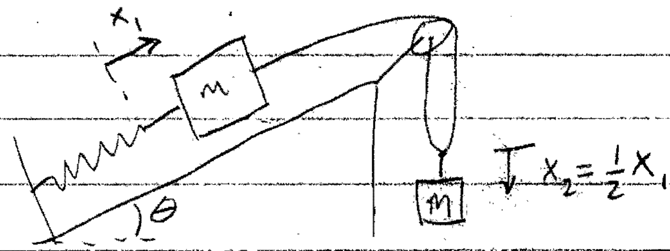


Matching components of $\underset{\sim}{a}$ and $\underset{\sim}{F}$ $\perp$ to $\underset{\sim}{r}$ we have

(+3) $$ma_{\perp} = mL\frac{d^2\theta}{dt^2} = -mg\sin\theta = F_{\perp}$$

Taking $\theta \ll 1$ so $\sin\theta \sim \theta$ and rearranging we get

(+2) $$\boxed{\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0}$$

2. 

Take $x_1 = 0$ (spring at unstretched length) as a reference position. At any instant, the total energy is the kinetic plus potential of the entire system (two blocks plus spring):

(+2) $$E = \left(\tfrac{1}{2}mv_1^2 + mgy_1 + \tfrac{1}{2}kx_1^2\right) + \left(\tfrac{1}{2}mv_2^2 + mgy_2\right)$$

The height $y_1 = x_1 \sin\theta$ and $y_2 = -\dfrac{x_1}{2}$ important!!!

The velocities are related since $x_2 = \frac{1}{2}x_1 \Rightarrow v_2 = \frac{1}{2}v_1$.

Plugging in,

(+2) $$E = \tfrac{1}{2}mv_1^2 + mgx_1\sin\theta + \tfrac{1}{2}kx_1^2 + \tfrac{1}{8}mv_1^2 - \tfrac{mg}{2}x_1$$

Energy is conserved, so $dE/dt = 0$. Taking the derivative, we get

(+3) $$\frac{dE}{dt} = \frac{1}{2}m2\cancel{v_1}\frac{dv_1}{dt} + mg\frac{\cancel{dx_1}}{\cancel{dt}}\sin\theta + kx_1\frac{\cancel{dx_1}}{\cancel{dt}} + \frac{1}{4}m\cancel{v_1}\frac{dv_1}{dt} - \frac{mg}{2}\frac{\cancel{dx_1}}{\cancel{dt}} = 0$$

Since $v_1 = dx_1/dt$, this term can be cancelled. Collecting terms, we have

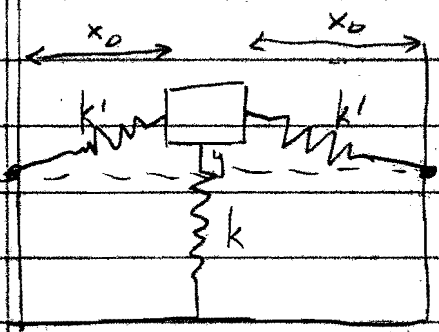
(+2) $$\frac{5}{4}m_1\frac{dv_1}{dt} + kx_1 + \left(mg\sin\theta - \tfrac{1}{2}mg\right) = 0$$

Now $dv_1/dt = d^2x_1/dt^2$ and we can define $y = x_1 - (mg\sin\theta - \frac{1}{2}mg)$

(+1) and note $d^2y/dt^2 = d^2x_1/dt^2$ to get $\boxed{\frac{5}{4}m_1\frac{d^2y}{dt^2} + ky = 0}$

3. VIBRATION ISOLATION

(+1)

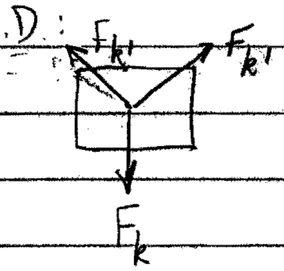


a. In the initial position, not shown,

$$P = -k'(x_o - L_o)$$

so $$\frac{P - k'L_o}{k'} = x_o$$

(+2) b. F.B.D:



Forces due to spring k and the two k' springs.

(+2) c. $F_k = -ky$ (y positive upward)

$$F_{k'} = -k'\left(\left(x_o^2 + y^2\right)^{1/2} - L_o\right)$$

d. Summing the force contributions in the y direction, we have

$$\Sigma F_y = F_k + 2F_{k'} \sin\theta = m\, d^2y/dt^2$$

with $$\sin\theta = \frac{y}{\left(x_o^2 + y^2\right)^{1/2}}$$

(+3) So,

$$-ky - 2k'\left(\left(x_o^2 + y^2\right)^{1/2} - L_o\right)\frac{y}{\left(x_o^2 + y^2\right)^{1/2}} = m\, d^2y/dt^2.$$

e. Assuming $y \ll x_o$, proceed carefully. $y \ll x_o$ means that we can neglect terms y/x_o relative to 1, or terms of $(y/x_o)^2$ relative to (y/x_o), etc. It does not mean set $y = 0$

In the equation of motion, we have

$$\left(x_0^2+y^2\right)^{1/2} = x_0\left(1+y^2/x_0^2\right)^{1/2} \approx x_0 \quad \text{since } \left(y/x_0\right)^2 \ll 1$$

So,

(+1)
$$\boxed{-ky - 2k'(x_0-L_0)\frac{y}{x_0} = m\,d^2y/dt^2}$$

f. The effective spring constant is the coefficient of the y term in the equation of motion:

$$\text{E.O.M.:} \quad m\frac{d^2y}{dt^2} + \left(k + 2k'\frac{(x_0-L_0)}{x_0}\right)y = 0$$

(+3)
$$k_{eff} = k + 2k'\frac{(x_0-L_0)}{x_0}$$

Substituting P, we have

$$k_{eff} = k - \frac{2P}{\frac{P-k'L_0}{k'}} = k - \frac{2k'P}{P-k'L_0}$$

g. $k_{eff} = 0 \Rightarrow k = 2k'P/P-k'L_0$

Solving for P gives

(+3)
$$\boxed{P = \frac{-kL_0}{2-k'/k}}$$

We can use any k' and L_0, within practical limits, and simply adjust P to this value to get $k_{eff}=0$

4. Molecular Potential $V(r) = D\left(1-e^{-\alpha(r-r_0)}\right)^2$ (5)

a. The force is $F(r) = -\frac{dV}{dr}$.

So

$$F(r) = -2D\left(1-e^{-\alpha(r-r_0)}\right)\left(-\alpha e^{-\alpha(r-r_0)}\right)$$

(+2)

When $r=r_0$, $F(r_0) = -2D(1-1)(-\alpha e^{1}) = 0$

so r_0 is the "equilibrium" or zero-force separation between the atoms.

b. To get the effective spring constant, we want to linearize $F(r)$ around r_0. Or, in terms of $V(r)$, we write

$$V(r) \approx V(r_0) + \underbrace{\left.\frac{\partial V}{\partial r}\right|_{r_0}}_{F(r_0)=0}(r-r_0) + \frac{1}{2}\left.\frac{\partial^2 V}{\partial r^2}\right|_{r_0}(r-r_0)^2$$

and so

$$F(r) = -\frac{dV}{dr} \approx -\left.\frac{\partial^2 V}{\partial r^2}\right|_{r_0}(r-r_0)$$

$$\Rightarrow k_{eff} = \left.\frac{d^2V}{dr^2}\right|_{r_0}$$ as was shown in class.

So

$$\frac{d^2V}{dr^2} = -\frac{dF(r)}{dr} = \frac{d}{dr}\left[2D\alpha\left(1-e^{-\alpha(r-r_0)}\right)e^{-\alpha(r-r_0)}\right]$$

(+1)

$$= \frac{d}{dr}\left[2D\alpha\left(e^{-\alpha(r-r_0)} - e^{-2\alpha(r-r_0)}\right)\right]$$

$$= 2D\alpha\left[-\alpha e^{-\alpha(r-r_0)} - (-2\alpha)e^{-\alpha(r-r_0)}\right]$$

(+2) $k_{eff} = \left.\frac{d^2V}{dr^2}\right|_{r_0} = 2D\alpha^2 = 2\times 4.08\,eV\,(.9Å^{-1})^2$

(+2)

$$= \boxed{6.61\ \frac{eV}{Å^2}} = \frac{6.61\times1.6\times10^{-19}\ N\text{-}m}{(10^{-10}m)^2} = \boxed{105.75\ \frac{N}{M}}$$

5 1/2

4. (cont.)

The natural frequency is

$$\omega_n = \sqrt{\frac{k_{eff}}{M}}$$

Here, we take $M = \frac{1}{2}(M_o + M_c) = \frac{1}{2}(28\ \text{amu})$

1 amu $= 1.67 \times 10^{-27}$ kg so

(+2) $$\omega_n = \sqrt{\frac{105.75\ \text{N/m}}{14 \times 1.67 \times 10^{-27}\ \text{kg}}} = 6.73 \times 10^{13}\ \frac{\text{rad}}{\text{s}}$$

units: $\sqrt{\frac{N/m}{kg}} = \sqrt{\frac{(kg\text{-}m/s^2)/m}{kg}} = \frac{1}{s}$ ✓

This is a very high frequency by engineering standards but is comparable to the frequencies of infrared radiation, which relates to greenhouse gases and global warming (see next week's HW!!)

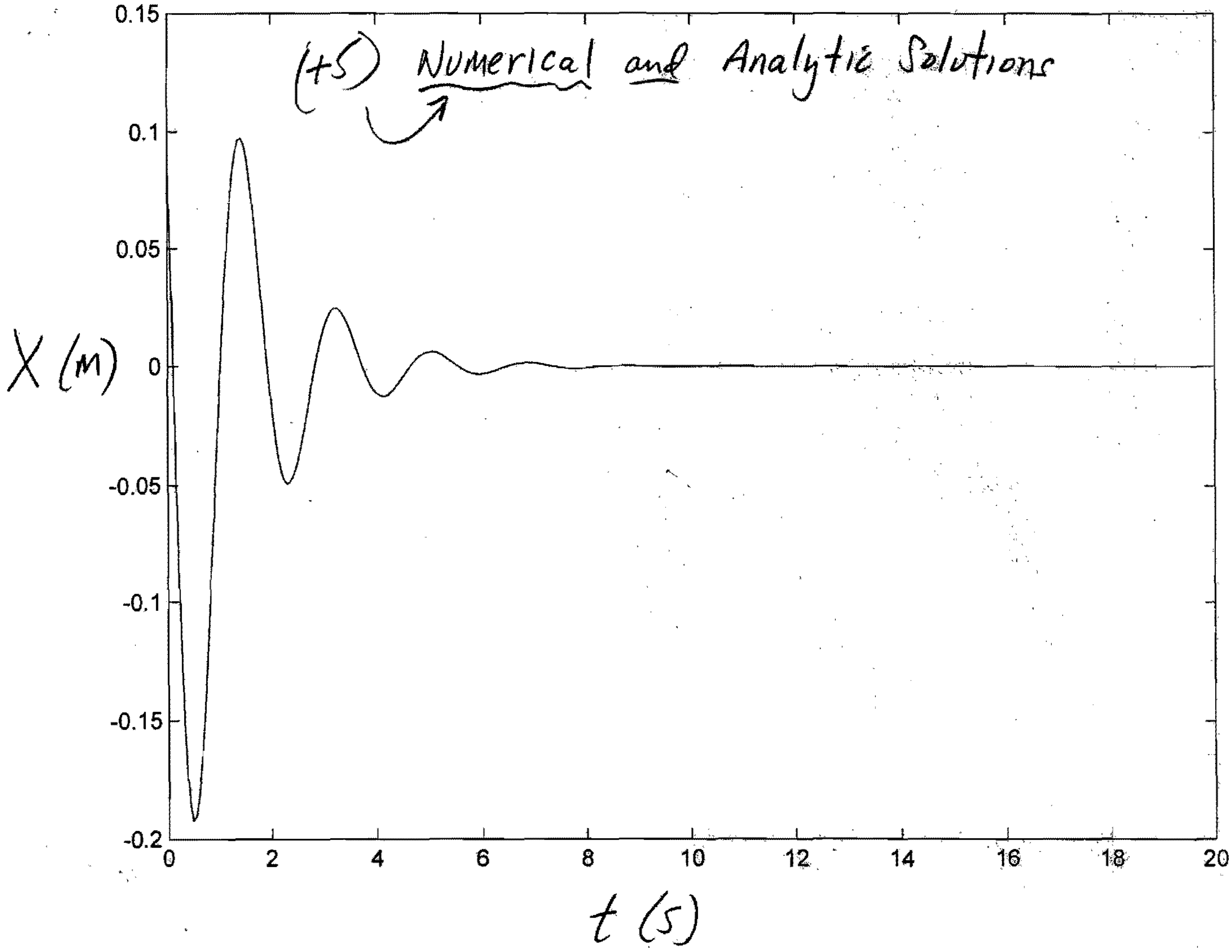
5. See the plot below.

The analytic solution is as follows:

(+1) 1. Calculate $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{25\ N/m}{2\ kg}} = 3.54\ \frac{rad}{s}$

(+1) 2. Calculate $\zeta = \frac{c}{2m\omega_n} = \frac{3\ N\text{-}s/m}{2(2kg)(3.54\frac{1}{s})} = 0.212 \Rightarrow \zeta < 1$ UNDERDAMPED

(+0) 3. Write down general solution: $x(t) = Xe^{-\zeta\omega_n t}\sin(\omega_d t + \psi)$



4. Apply initial conditions:

$x(t=0) = 0.1\ m \Rightarrow x(0) = \boxed{X\sin\psi = 0.1m}$

(+3) $\frac{dx}{dt}(t=0) = -1.0\ \frac{m}{s} \Rightarrow \frac{dx}{dt}(t=0) = X\left(-\zeta\omega_n e^{-\zeta\omega_n t}\sin(\omega_d t + \psi) + e^{-\zeta\omega_n t}\omega_d\cos(\omega_d t + \psi)\right)$

at $t=0$ $= \boxed{X(-\zeta\omega_n\sin\psi + \omega_d\cos\psi) = -1.0 m/s}$

See next page....

5. (continued)

Dividing, to eliminate X, we get

$$\frac{dx/dt\,(t=0)}{x\,(t=0)} = \frac{-\zeta\omega_n \sin\psi + \omega_d \cos\psi}{\sin\psi} = -\zeta\omega_n + \frac{\omega_d}{\tan\psi} = \frac{-1.0\,\text{m/s}}{0.1\,\text{m}}$$

$\omega_d = \omega_n\sqrt{1-\zeta^2} = 3.46\,\frac{1}{s}$ so we can solve for $\tan\psi$ as

$$\tan\psi = -.374 \Rightarrow \psi = -.358\text{ rad} \Rightarrow \sin\psi = -.350$$

so $X = \dfrac{0.1\,\text{m}}{\sin\psi} = -0.285\,\text{m}$

$$\Rightarrow \boxed{x(t) = (-0.285\,\text{m})\, e^{-0.75\,t/s} \sin(3.46\,t/s - .358)}$$

6. During operation at constant T, the thruster + stand compresses the spring to a distance satisfying

$$T - kx_o = 0 \quad \text{or} \quad x_o = T/k$$

This x_o is the initial position of the system when the thrust is turned off. The velocity at this instant is zero.

(+2) To have $x_o < 1\text{ ft}$ requires $T/k < 1\text{ ft} \Rightarrow \underline{\underline{k > 10{,}000\text{ lb/ft}}}$

(+1) To return to equilibrium as fast as possible requires a design for which $\zeta = 1$ (Critical damping). For this spring, mass, dashpot system we have

$$\zeta = \frac{c}{2m\omega_n}, \quad \omega_n = \sqrt{k/m}$$

(+1) For $k = 10000\text{ lb/ft}$, $mg = 1500\text{ lb}$, we have $\underline{\underline{\omega_n = 14.65\ \tfrac{\text{rad}}{\text{s}}}}$

(+2) So $\zeta = 1$ requires $c = 2m\omega_n = 2\dfrac{(1500\text{ lb})}{32.2\text{ ft/s}^2}\, 14.65\tfrac{\text{rad}}{\text{s}} = \underline{\underline{1365\text{ lb-ft/s}}}$

Optional +2 extra

The motion is then described by

$$x(t) = (C_1 + C_2 t)e^{-\omega_n t}$$

Applying initial conditions,

$$x(0) = \underline{\underline{C_1}} = x_o$$

$$\dot{x}(0) = \left.\left(-\omega_n C_1 e^{-\omega_n t} + C_2 e^{-\omega_n t} - \omega_n C_2 \cancel{t}^{\,0} e^{-\omega_n t}\right)\right|_{t=0} = 0$$

so

$$-\omega_n C_1 + C_2 = 0 \Rightarrow C_2 = \omega_n C_1 = \omega_n x_o$$

So,

$$x(t) = x_o(1 + \omega_n t)e^{-\omega_n t}$$

7. Ear Infections

From the data shown, the damped period is

(+2) $$\tau_d \approx \frac{0.18\,s}{9\text{ cycles}} = \underline{\underline{0.02\,s}}$$

so

(+2) $$\omega_d = \frac{2\pi}{\tau_d} = 314.2\ \frac{rad}{s}$$

To get ζ, we use the logarithmic decrement. It is not so accurate to look at two successive peaks, so let's look at first peak and ninth peak, at t_1 and $t_1 + 9\tau_d$. Then

$$\frac{X_1}{X_9} = \frac{e^{-\zeta\omega_n t_1}\ \cancel{\sin(\omega_d t_1 + \phi)}}{e^{-\zeta\omega_n(t_1+9\tau_d)}\ \cancel{\sin(\omega_d(t_1+9\tau_d)+\phi)}} = e^{9\zeta\omega_n\tau_d}$$

So,

$$\ln\frac{X_1}{X_9} = 9\zeta\omega_n\tau_d = 9\zeta\omega_n\frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{18\pi\zeta}{\sqrt{1-\zeta^2}}$$

From the data, $X_1 \approx 3.1\times10^{-4}$, $X_9 \approx 1.3\times10^{-4}$ so

(+3) $$\ln\frac{X_1}{X_9} = 0.869 = \frac{18\pi\zeta}{\sqrt{1-\zeta^2}} \Rightarrow \boxed{\zeta = 0.015}$$

For the ear drum modeled as a mass/spring/dashpot,

$$\zeta = \frac{c}{2m\omega_n} \quad \text{so} \quad c = 2m\omega_n\zeta \qquad \omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} \approx \omega_d = 314.2\ \frac{rad}{s}$$

(+2) Using $m = 0.1\,g$ (10^{-4} kg) we have $\boxed{c = 0.00094\ N\text{-}s/m}$