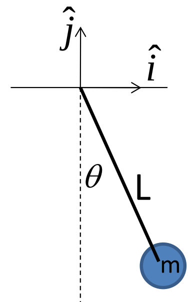
EN 4
Dynamics and Vibrations

Homework 5: Free and Damped Vibrations

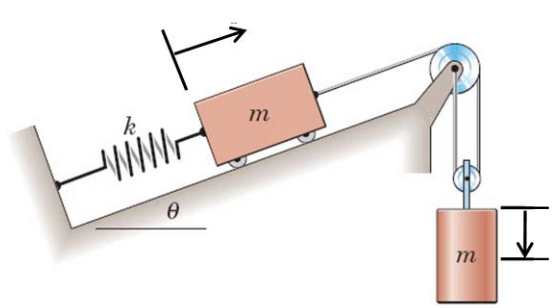
1. The Pendulum (revisited): It was shown in class using conservation of energy that the equation of motion for a pendulum, for small amplitudes of vibration $\theta << 1$, is

$$\frac{d^2\theta}{dt^2} + \sqrt{\frac{g}{L}}\ \theta = 0$$



Derive this same result as follows. First, apply $\underset{\sim}{F} = m\underset{\sim}{a}$ using the strategy developed in the first part of this course (Define position vector; determine acceleration vector; draw Free Body Diagram, etc.) to get the equations of motion in the i and j coordinates. Second, use those results to consider the motion *perpendicular to the pendulum arm* at any instant and thus get the above equation of motion for the angle θ as a function of time.

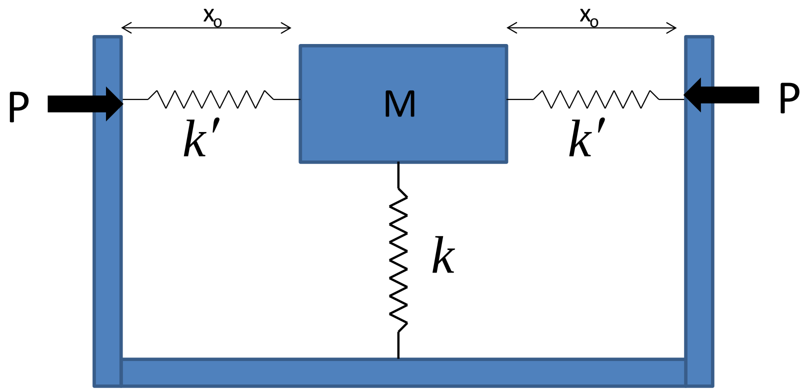
2. Conservation of Energy: Determine the equation of motion for the position x_1 of the left mass as a function of time *using energy conservation*. The answer, found using $\underset{\sim}{F} = m\underset{\sim}{a}$, is given in the notes for Vibrations Lecture #1.



3. Design of a Vibration Isolation System:

We have shown that a mass m connected by a spring (spring constant k) to a supporting structure will vibrate at a natural frequency $\omega_n = \sqrt{k/m}$. In the Barus-Holley mini-quake, we have seen that if the support is oscillating with a frequency $\omega = \omega_n$ then the vibration amplitude can be increased dramatically (resonance). For many scientific experiments, such as those going on in Barus and Holley, vibrations of the building, if transmitted to experiments sitting on table tops, ruin the experiments. To solve this problem, a Vibration Isolation System, i.e. a configuration that protects the experiment from the underlying vibrations, is used. You will design a system that is currently marketed commercially by "Minus k Technologies" (http://www.minusk.com/), which was shown in class.

The idea is to modify the system so that the *effective* spring constant acting on the support is zero. In other words, for a system with a normal spring constant *k*, we need to devise a system that adds -*k* so that the net contribution is $k_{eff} = k + (-k) = 0$. The natural frequency is then 0 and the mass cannot undergo resonance. The design is shown below:



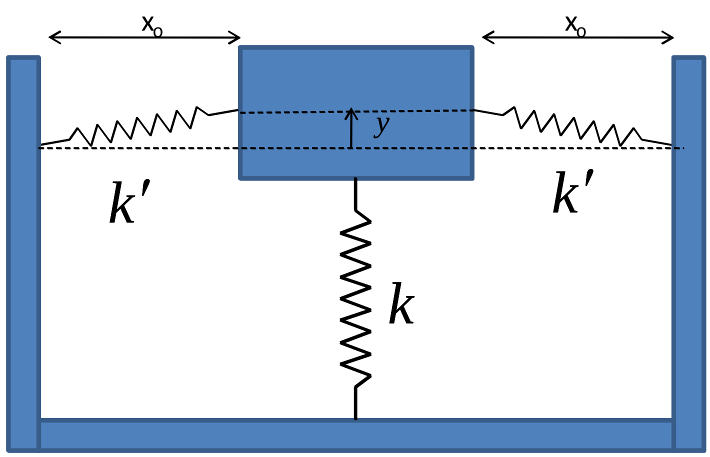
The mass M is supported by a vertical spring of spring constant *k*. The unstretched length of this spring is irrelevant to the analysis. When the mass is in static equilibrium, two lateral springs, each having spring constant *k'*, are added on the sides. These springs are put under an initial compression of force P. Thus, the initial length x_o is *smaller* than the unstretched length L_o of the lateral springs. The lateral springs are connected to the sides and mass by pin joints, and so are free to rotate.

The goal of this problem is to choose P in terms of *k*, *k'*, and L_o so that the natural frequency of vertical vibrations is *zero*, for small amplitude vibrations.

Proceed as follows:

a. Determine the initial compressed length x_o in terms of P, L_o, and *k'*.

b. For a small vertical displacement y of the mass, draw the Free Body Diagram. Since this displacement is *relative to* static equilibrium, which is determined by gravity, we can neglect gravity in considering the F.B.D.



c. Express the magnitude of the forces in each of the springs in terms of x_o, L_o, k', k, and y.

d. Use $\underset{\sim}{F} = m\underset{\sim}{a}$ to obtain an equation of motion for the vertical displacement y.

e. Assume that $y << x_o$ (small displacement assumption) and obtain a simplified equation of motion.

f. Identify the *effective* spring constant k_{eff} for the system in terms of x_o, L_o, *k'*, *k*.

g. Determine the lateral force P needed to make $k_{eff}=0$.

4. Effective Spring Constant for a Molecule

A common model for the interactions between atoms is the Morse potential, which has the form

$$V(r) = D(1 - e^{-\alpha(r - r_o)})^2$$

where *r* is the separation between two atoms, r_o is the equilibrium (zero force) distance between the two atoms, D is an energy parameter, and α (units of m^{-1}) is a measure of how quickly the bond energy increases as the distance changes away from the equilibrium distance.

For carbon dioxide, each C-O bond can be described by the Morse potential with D=4.08eV (1eV=1.6×10^{-19}J), $r_o = 1.5A$ (1 Angstrom = 10^{-10}m), and $\alpha = 0.9A^{-1}$.

a. Show that r_o is indeed the equilibrium molecular spacing (i.e. the point of zero force).

b. Derive the effective "spring constant" k for linear vibration of a C-O bond for small displacements $|r - r_o| << r_o$ in terms of any of the parameters D, r_o, and α, and calculate a numerical value.

c. Taking the mass as the average of the masses m_O and m_C for Oxygen and Carbon, (found easily from reference sources), determine the natural frequency of vibration of the C-O molecular bond.

5. Basic Damped Vibrations (or, every HW set needs a little MATLAB)

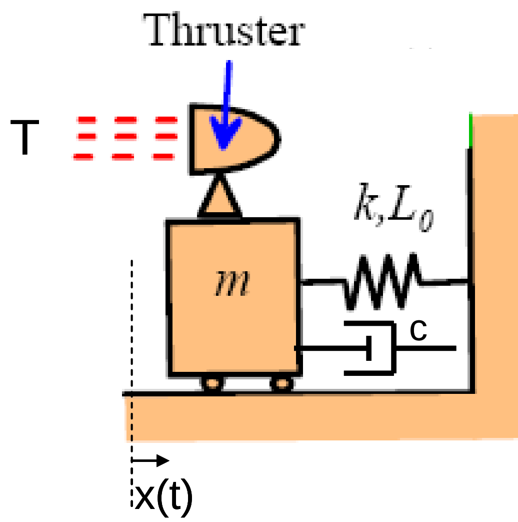
The generic equation of motion for a damped vibrating system is

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

where m is the "mass", c is the viscous damping coefficient, and k is the spring constant. Even though we can solve this equation analytically, develop a basic MATLAB code using the ODE solver to compute *x(t)* given input values for m, c, and k, and initial position x_o and velocity v_o.

Verify your code by showing a graph of the numerical MATLAB solution and the exact analytic solution developed in class for m=2 kg, k=25 N/m, c= 3 N-s/m, and x_o=0.1 m, v_o= -1.0 m/s. To use the analytic solution, you need to determine the natural frequency of vibration, the amplitude, and the phase angle.

6. Engine Test Stand Recoil



An engine test stand used to measure the thrust of a jet engine is built to include a shock absorber to minimize recoil and vibrations when the thrust is turned on or off. Consider the system to be running with constant thrust T=10,000 lbs, and then to have the thrust turned off abruptly.

Design a system (i.e. choose values for k and c) so that the engine + test stand (total weight = 1500 lbs) has a deflection of less than 1 ft. and returns to equilibrium as fast as possible when the thrust is turned off. The spring is unstretched when x(t)=0.

7. Testing for Ear Infections

To test for fluid behind the ear drum in the middle ear, an impulse test can be done wherein pressure pulse is used to impart an initial velocity to the ear drum, which is initially at rest. The response of the ear drum, for instance the deflection of the center of the ear drum, is then measured, as shown in the graph below.

Determine the effective viscous damping coefficient c, the magnitude of which is an indication of fluid behind the ear drum caused infection. In reality, the physical situation is much more complicated but the test (a Tympanogram) is a static pressure test.