EN 4 HW 6 SOLUTIONS

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1. From last week's MATLAB, we had $M = 2\,kg$, $k = 25\,N/m$, $C = 3\,N\text{-}s/m$, $x_0 = 0.1\,m$, and $V_0 = -1.0\,m/s$.

For a forced vibration, $F(t) = F_0 \sin \omega t$, we need to specify F_0 and examine the steady-state amplitude of vibration X. We normalize X by the static deflection caused by a force F_0, i.e. $X_0 = F_0/k$.

In my MATLAB, I used $F_0 = 1N$ so $X_0 = .04\,m$.

For various ω, the response is:

ω (rad/s)	X/X_0
0.5	~1.0
1.0	~1.1
1.5	~1.2
2.0	~1.4
2.5	~1.75
3.0	~2.2
3.5	~2.3
4.0	~1.75
4.5	~1.2
5.0	~0.8
6.0	~0.5
7.0	~0.3

* Natural frequency is $\omega_n = 3.5355$

* Damping coeff is $\zeta = 0.2121$

* Amplitude at resonance should by $\frac{X}{X_0} \sim \frac{1}{2\zeta} = 2.36$

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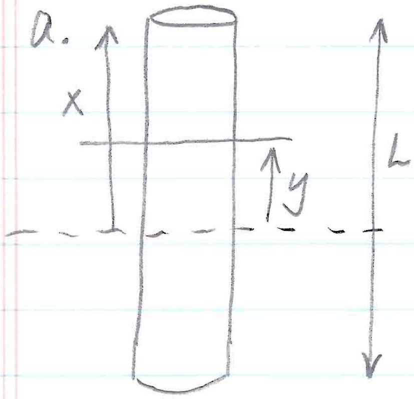
2. a.



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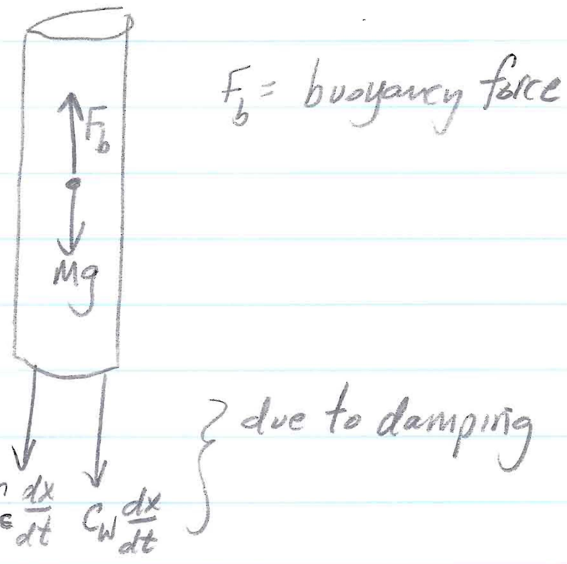

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+2 The buoyancy force is

$$F_b = \underbrace{\rho_w A\left(L-(x-y)\right)}_{\text{mass of displaced water}} g$$

+1 The force of gravity is

$$F_g = mg = \rho_b A L g \quad [m = \rho_b A L]$$

b. Summing forces in the vertical direction (positive up),

$$\sum F = F_b - mg - (C_w + C_c)\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

Using part a.,

$$\rho_w A\left[L-(x-y)\right]g - \rho_b A L g - (C_w + C_c)\frac{dx}{dt} = \rho_b A L \frac{d^2x}{dt^2}$$

Rearranging,

$$\frac{d^2x}{dt^2} + \frac{C_w + C_c}{\rho_b A L}\frac{dx}{dt} - \frac{\rho_w A g\left(L-(x-y)\right)}{\rho_b A L} + g = 0$$

Moving y, which is some y(t), to the right hand side,

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$$\boxed{\frac{d^2x}{dt^2} + \frac{C_w + C_c}{\rho_b A L}\frac{dx}{dt} - \frac{\rho_w g(L-x)}{\rho_b L} - g = \frac{\rho_w g}{\rho_b L}y}$$

+1 c. The static position is obtained by setting $\frac{d^2x}{dt^2} = \frac{dx}{dt} = 0$ and $y = 0$ (no motion). Then,

$$-\frac{\rho_w g}{\rho_b L}(L - x_o) - g = 0 \Rightarrow \boxed{x_o = L(1 - \rho_b/\rho_w)}$$

+2 d. Let $z = x - x_o$. Then $\frac{d^2z}{dt^2} = \frac{d^2x}{dt^2}$, $\frac{dz}{dt} = \frac{dx}{dt}$, as the constant terms in the E.O.M. drop out:

$$\boxed{\frac{d^2z}{dt^2} + \frac{c_w + c_c}{\rho_b A L}\frac{dz}{dt} + \frac{\rho_w g}{\rho_b L} z = \frac{\rho_w g}{\rho_b L} y}$$

+1 e. $y = y_{max} \sin \omega t$

f. $$\frac{d^2z}{dt^2} + \frac{c_w + c_c}{\rho_b A L}\frac{dz}{dt} + \frac{\rho_w g}{\rho_b L} z = \frac{\rho_w g y_{max}}{\rho_b L} \sin \omega t$$

+2 So, $\boxed{\omega_n = \sqrt{\frac{\rho_w g}{\rho_b L}}}$ (somewhat like a pendulum!)

+1 Then $2\zeta\omega_n = \frac{c_w + c_w}{\rho_b A L} \Rightarrow \zeta = \frac{c_c + c_w}{2\rho_b A L}\sqrt{\frac{\rho_b L}{\rho_w g}} = \boxed{\frac{c_c + c_w}{2A\sqrt{\rho_b \rho_w g L}}}$

+1 and $\frac{F_o}{m} = \frac{\rho_w g y_{max}}{\rho_b L}$ so $F_o = \cancel{\rho_b} A \cancel{L} \cdot \frac{\rho_w g y_{max}}{\cancel{\rho_b L}} = \boxed{\rho_w g A y_{max}}$

g. $z = \frac{F_o}{m\omega_n^2} M$ in general, where M is the "Magnification factor"

+2 But $\frac{F_o}{m\omega_n^2} = \frac{\rho_w g y_{max}}{\rho_b L}\left(\frac{\rho_b L}{\rho_w g}\right) = y_{max}$

So $z = y_{max} M$ Buoy motion amplified by M over wave motion.

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h. For small ζ, we want $\omega_n = \omega$ (Resonance) for a given wave frequency ω. So

$$\sqrt{\frac{\rho_w g}{\rho_b L}} = \omega \quad \text{or} \quad \boxed{\rho_b L = \frac{\rho_w g}{\omega^2}}$$

Any combo of ρ_b, L (within limits not considered here) that satisfies the above equation will match the buoy response to the wave motion.

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i. At resonance, $M \sim \frac{1}{2\zeta}$ so $\boxed{Z_{max} = \frac{y_{max}}{2\zeta}}$

j. Power = (Force)·(Velocity).
The force in the electrical dashpot is $-c_c \underset{\sim}{V}$. So, the power is

$$P = -c_c \underset{\sim}{V} \cdot \underset{\sim}{V} = -c_c V^2$$

(minus indicates energy lost from the buoy motion).

But

$$V = \frac{dZ}{dt} = Z_{max}\,\omega\cos(\omega t + \phi)$$

so

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$$\boxed{P = -c_c Z_{max}^2 \omega^2 \cos^2(\omega t + \phi)}$$

Averaging over one period of vibration ($\tau = \frac{2\pi}{\omega}$) we have

$$\overline{P} = \frac{1}{\tau}\int_0^\tau P\,dt = -c_c Z_{max}^2 \omega^2 \frac{1}{\tau}\int_0^\tau \cos^2(\omega t + \phi)\,dt$$

(Integral: $\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \cos^2(\omega t + \phi)\,dt = \frac{1}{2}$; do it yourself for practice)

so

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$$\overline{P} = \frac{1}{2} c_c Z_{max}^2 \omega^2 = \frac{c_c y_{max}^2 \omega^2}{8\zeta^2}$$

[drop "−" sign now]

j. (cont). Since we are at resonance, $\omega = \omega_n$. And we have ζ from before, so

$$\bar{P} = \frac{c_c\, y_{max}^2}{8\,(c_c + c_w)^2}\; 4A^2\left(\cancel{\rho_b}\rho_w g \cancel{A}\right)\cdot \frac{\rho_w g}{\cancel{\rho_b A}}$$

or

+1 $$\bar{P} = \frac{y_{max}^2 A^2 \rho_w^2 g^2}{2}\; \frac{c_c}{(c_c + c_w)^2}$$

k. We now want to maximize $\bar{P}$ with respect to c_c.
So, find c_c where

$$\frac{d\bar{P}}{dc_c} = 0 \Rightarrow \frac{1}{(c_c+c_w)^2} - \frac{2c_c}{(c_c+c_w)^3} = 0$$

+2 $$\Rightarrow c_c + c_w = 2c_c \Rightarrow \boxed{c_c = c_w}$$

For $c_c = c_w$, $\dfrac{c_2}{(c_c+c_w)^2} = \dfrac{1}{4c_w}$

Thus,

$$\boxed{\bar{P} = \frac{y_{max}^2 A^2 \rho_w^2 g^2}{8\, c_w}}$$

Check units: Power $= F\,v = kg\text{-}\frac{m}{s^2}\cdot\frac{m}{s} = kg\text{-}m^2/s^3$

$$\frac{y_{max}^2 A^2 \rho_w^2 g^2}{c_w} = \frac{(m^2)(m^4)\left(kg/m^3\right)^2\left(m/s^2\right)^2}{\left(kg\frac{m}{s^2}\cdot s/m\right)} = kg\text{-}m^2/s^3 \quad ✓$$

3. This is a "Rotor Excitation" problem.
To solve, we need to know ω_n, ζ, the mass imbalance Δm, the eccentricity e, and the operating frequency.
Here, we are given

$$\Delta m = 0.5\,kg,\quad e = 0.3\,m,\quad \zeta = 0 \text{ (no damping)}.$$
$$m = 35\,kg \quad\text{and}\quad \omega \Leftrightarrow 275\text{ RPM}$$

From the static info (35 kg compresses four springs a distance of 0.01m) we have

$$mg = 4k\delta \qquad \delta = .01\,m$$

+1 so

$$k = \frac{mg}{4\delta} \Rightarrow \omega_n = \sqrt{\frac{4k}{m}} = \sqrt{\frac{g}{\delta}} = 31.32\,\frac{rad}{s}.$$

With 275 RPM corresponding to

+1 $$\omega = 2\pi \times \frac{275\,rad}{60\,s} = 28.8\,\frac{rad}{s},$$

the steady-state amplitude of vibration X satisfies

+1 $$\frac{X}{\left(\frac{\Delta m}{m}e\right)} = \left(\frac{\omega}{\omega_n}\right)^2 M \qquad M = \frac{1}{\sqrt{\left(1-\left(\omega/\omega_n\right)^2\right)^2 + \left(2\cancel{\zeta}\frac{\omega}{\omega_n}\right)^2}}$$

Plugging in,

$$X = \frac{\Delta m\, e}{m}\left(\frac{\omega}{\omega_n}\right)^2 \frac{1}{\left|1-\left(\omega/\omega_n\right)^2\right|} = \frac{0.5\,kg}{35\,kg}\,0.3\,m\left(\frac{28.8}{31.32}\right)^2 \frac{1}{1-\left(\frac{28.8}{31.32}\right)^2}$$

+2 $$\boxed{X = 0.023\,m = 2.3\,mm}$$

2.3× the static deflection but not too big, perhaps.

4. a. Governing equation is

$$L\frac{dI}{dt} + RI + \frac{1}{C}q = V_o \sin \omega t$$

Subbing in $I = dq/dt$ yields

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V_o \sin \omega t$$

+2 or $$\boxed{\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = \frac{V_o}{L}\sin \omega t}$$

+1 b. $\omega_n = \sqrt{\frac{1}{LC}}$ (square root of coefficient multiplying q)

c. At steady-state, we have

$$q(t) = Q\sin(\omega t + \phi)$$

where

$$Q = \frac{(V_o/L)}{\omega_n^2}M \quad \text{since } "\frac{F_o}{m}" = \frac{V_o}{L} \text{ here}$$

So,

$$I(t) = Q\omega \cos(\omega t + \phi) = I\cos(\omega t + \phi)$$

and at $\omega = \omega_n$, $M \approx 1/2\zeta$ so

$$I = Q\omega_n M = \frac{V_o/L}{\omega_n^2}\omega_n \cdot \frac{1}{2\zeta}$$

From a., $2\zeta\omega_n = R/L$ so $\frac{1}{2\zeta} = \omega_n L/R$. Hence

+2 $$I = \frac{V_o/\cancel{L}}{\cancel{\omega_n}} \cdot \cancel{\omega_n}\cancel{L}/R = \boxed{\frac{V_o}{R} = I}$$

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5. a.

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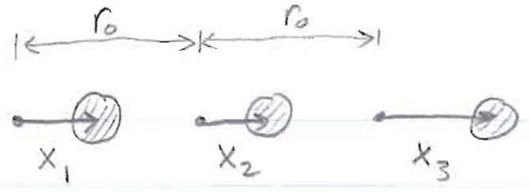

b. [assuming equal masses]

(1) $$m\frac{d^2x_1}{dt^2} = k(x_2-x_1)$$

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(2) $$m\frac{d^2x_2}{dt^2} = -k(x_2-x_1) + k(x_3-x_2)$$

(3) $$m\frac{d^2x_3}{dt^2} = -k(x_3-x_2)$$

c. Let $y = x_2 - x_1$ and $z = x_3 - x_2$.

Now, subtract Eq. 1 from Eq. 2 to get

$$m\frac{d^2(x_2-x_1)}{dt^2} = -2k(x_2-x_1) + k(x_3-x_2)$$

or

$$\boxed{m\frac{d^2y}{dt^2} = -2ky + kz}$$

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Subtracting Eq. 2 from Eq. 3 gives

$$m\frac{d^2(x_3-x_2)}{dt^2} = -2k(x_3-x_2) + k(x_2-x_1)$$

or

$$\boxed{m\frac{d^2z}{dt^2} = -2kz + ky}$$

d. (1) 3 atoms, 3 directions of motion $\Rightarrow$ 9 degrees of freedom

+1 (2) 3 atoms, 1 direction of motion $\Rightarrow$ 3 degrees of freedom

(3) $3-1 = 2$ degrees of freedom for vibrations, matching w/ c.

e. In matrix form,
$$m\frac{d^2}{dt^2}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$$

We want the eigenvalues λ^2 of the matrix, which are equal to the natural frequencies squared, ω_n^2.
We take the determinant:
$$\det\begin{vmatrix} -\frac{2k}{m}-\lambda^2 & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m}-\lambda^2 \end{vmatrix} = \left(-\frac{2k}{m}-\lambda^2\right)^2 - \left(\frac{k}{m}\right)^2 = 0$$
and solve for λ^2:
$$\lambda^2 = \pm\frac{k}{m} - \frac{2k}{m} = -\frac{k}{m}, \ -\frac{3k}{m}$$
so the natural frequencies are
$$\boxed{\omega_n = |\lambda| = \sqrt{\frac{k}{m}}, \sqrt{\frac{3k}{m}}}$$

+3

f. From last week, we already found $\sqrt{\frac{k}{m}} = 6.73\times 10^{13}\ \frac{rad}{s}$

so
$$\omega_n = 6.73\times 10^{13}\ \frac{1}{s} \ \text{ and } \ 11.66\times 10^{13}\ \frac{1}{s}$$

not a spring stiffness.

+1 The wavenumbers are $k = \frac{\omega_n}{c}$ [see Problem definitions]

and so
$$k = \frac{\omega_n}{3\times 10^8 \frac{m}{s}} = \boxed{1295\ cm^{-1},\ 2243\ cm^{-1} = k}$$

[Close to actual numbers quoted in HW]

This is in the INFRARED (as discussed in class)

g. If O atoms move oppositely while C is fixed, i.e.

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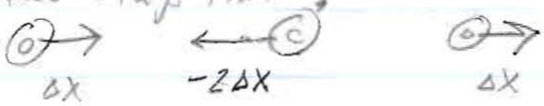
Then the O atoms are like a mass on a spring so

$$\omega_n = \sqrt{\frac{k}{M}} = 6.73\times10^{13}\,\frac{1}{s} \quad (\text{wavenumber } 1295\ \text{cm}^{-1})$$

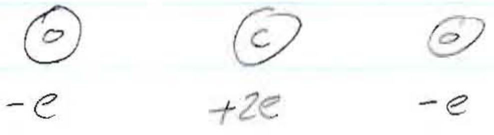
If the O atoms move in the same direction while C

+1 moves in the other direction, by twice the amount so the center of mass stays fixed:



then the forces on the O atoms are $3k\,\Delta X$ and one could surmise that the vibration frequency would be $\omega_n = \sqrt{\frac{3k}{m}}$ (wavenumber $2243\ \text{cm}^{-1}$)

These motions can be determined by finding the "eigenvectors" of the eigenvalues λ^2, as done in class for a slightly different system.

h. 

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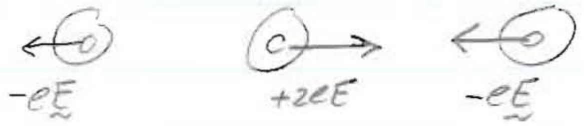
F.B.D.: 

Consistent with the $2243\ \text{cm}^{-1}$ mode of vibration

+1 i $\omega = \omega_n$ for $2243\ \text{cm}^{-1}$, i.e. $\underline{\omega = 11.66\times10^{13}\ 1/s}$

+1 j. O_2, N_2 are symmetric. An $\underset{\sim}{E}$ field cannot force them to vibrate. CH_4 has 2 different atom types and could vibrate (it is <u>worse</u> than CO_2!!)