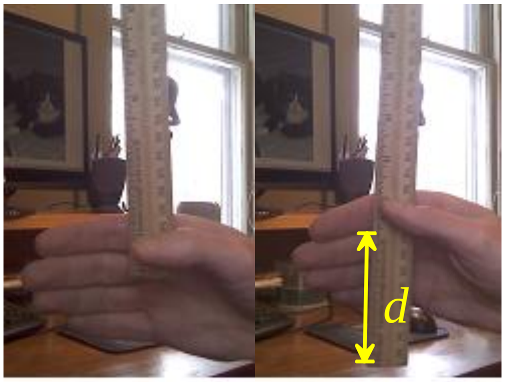
Division of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 2: Dynamics of Particles
Solutions. MAX SCORE 60 POINTS

1. The figure illustrates a simple method for measuring a person's reaction time. A ruler is suspended vertically between the test subject's fingers. The ruler is then dropped, and the test subject catches it by closing his or her fingers. The reaction time can be determined from the distance traveled by the ruler before the subject brings it to rest.



1.1 After the ruler is released, it drops vertically with constant acceleration of g=9.81 m/s^2. Write down an equation relating the distance d dropped by the ruler to the reaction time t

The ruler falls along a straight line with constant acceleration. The constant acceleration formulas give $y = y_0 + v_0 t + at^2/2$ where y_0, v_0, a are the initial position, velocity and acceleration, respectively. In this case $y_0 = v_0 = 0$, $a = g$, and y=d at time $t = t_r$. Thus

$$d = gt_r^2/2 \Rightarrow t_r = \sqrt{2d/g}$$

[2 POINTS]

1.2 Find a friend, and have him or her measure your reaction time using this approach (if you have no friends, the faculty or TAs will be happy to help you do the experiment. Some of them may even be willing to be your friend)

Typical reaction times are between 0.1 and 0.2s. A good solution will report the average of several tests.

[2 POINTS]

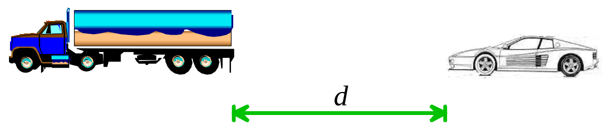
1.3 Repeat the experiment, but this time determine your reaction time while sending a text message on your cell-phone (we assume you can text with only one hand, as you would while driving a car…)

This can easily make reaction time exceed 0.5s and even up to 1s or so

[1 POINT]

1.4 **OPTIONAL – not for credit:** If your religion, state laws and safety considerations permit, you may be interested in measuring the influence of a few beverages of your choice on your reaction time. If you are unable to be a subject yourself, the TAs will probably be delighted to volunteer, provided that you supply their beverages.

2. Two vehicles traveling at speed V are separated by a distance d as shown in the figure. At time $t=0$ the leading vehicle strikes an obstacle and is brought to a complete standstill. The driver of the trailing vehicle takes time t_r to react, and then applies the brakes, decelerating her car at rate $\mathbf{a} = -0.8g\mathbf{i}$, where g is the acceleration due to gravity.



2.1 Find a formula for the minimum separation distance d that will prevent a collision.

The motion of the tailing vehicle can be divided into two parts

- For time $0 < t < t_r$ the vehicle travels at constant speed V and covers a distance $d = Vt_r$
- For time $t_r < t < t_2$ the vehicle has constant acceleration. During this period, the speed and distance traveled are given by the constant acceleration formulas

$$x = x_0 + v_0(t - t_0) + a(t - t_0)^2 / 2$$
$$v = v_0 + a(t - t_0)$$

with initial conditions $x_0 = Vt_r \quad v_0 = V, \quad t_0 = t_r$. We know that at time t_2 the speed of the car is zero, and the vehicle has traveled a distance d. Therefore

$$d = Vt_r + V(t_2 - t_r) - 0.8g(t_2 - t_r)^2 / 2$$
$$0 = V - 0.8g(t_2 - t_r)$$

These two equations can be solved for t_2 and d. Here's a MAPLE solution

```
> eq1 := d = V*tr+V*(t2-tr) - 0.8*g*(t2-tr)^2/2:
> eq2 := 0 = V-0.8*g*(t2-tr):
> solve({eq1,eq2},{d,t2});
```

$$\left\{ d = \frac{0.1250000000 \; V\,(5.\; V + 8.\; g\; tr)}{g},\; t2 = \frac{0.2500000000 \;(5.\; V + 4.\; g\; tr)}{g} \right\}$$

So $d = Vt_r + 0.625V^2 / g$

[3 POINTS]

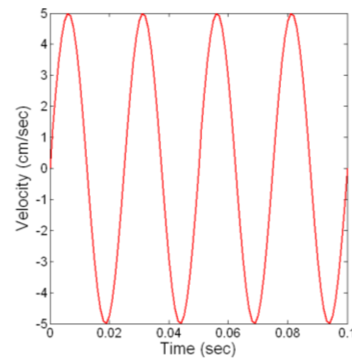
2.2 Estimate a value for d assuming that V=65MPH and using your answers to 1.2 and 1.3.

65mph is 29.06m/s. A stopping distance with a 0.2s reaction time is therefore 59.6m or 195 ft

With a 0.5s reaction time this increases to 68m or 223ft
With a 1s reaction time we get 82m or 269ft.

[2 POINTS]

3. The figure shows a vibration measurement from a velocity transducer. The vibration may be assumed to be harmonic.



Estimate

(a) The period of oscillation

There are 4 cycles in 0.1 sec, so the period is 0.025sec

(b) The angular frequency of oscillation

The angular frequency is related to the period by $\omega = 2\pi / T = 80\pi$ rad/s

(c) The amplitude of the velocity:

The amplitude is 5cm/s

(d) The amplitude of the displacement.

For simple harmonic motion we know that

$$x = X_0 \sin \omega t$$
$$v = \omega X_0 \cos \omega t = V_0 \cos \omega t$$
$$a = -\omega^2 X_0 \sin \omega t = -A_0 \sin \omega t$$

The amplitude of position, velocity and acceleration are $X_0, \omega X_0, \omega^2 X_0$, respectively

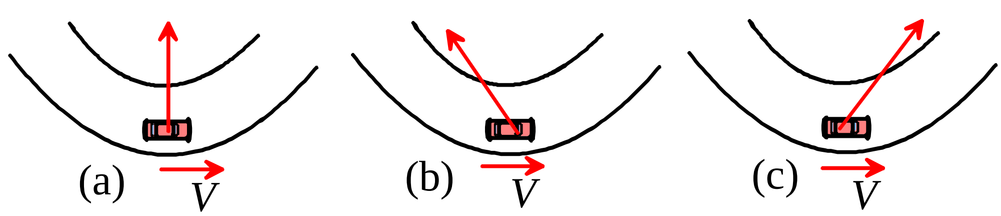
This means that the amplitude of the velocity is related to the amplitude of the displacement by

$$X_0 = V_0 / \omega = 1/(16\pi) = 0.0199 cm$$

(e) The amplitude of the acceleration.

The acceleration amplitude is given by $A_0 = \omega V_0 = 400\pi cm / s^2$

[5 POINTS – one for each part]



4. The figure shows a car that travels along a circular road.

In Fig 1(a), the car travels at constant speed.

In Fig 1(b), the driver is braking, and the car's speed is decreasing.

In Fig 1(c), the driver has her foot on the gas and the car's speed is increasing.

Draw an arrow on a copy of each of figures (a), (b), (c) to show the approximate direction of the car's acceleration vector.

Recall that:

Consider a particle that travels at speed V around a path with radius of curvature R. The acceleration vector can be constructed by adding two components:

- ***the component of acceleration tangent to the particle's path is equal to*** dV/dt
- ***The component of acceleration perpendicular to the path (towards the center of curvature) is equal to*** V^2/R.

This gives acceleration vectors as shown in the picture (for (a), $dV/dt = 0$, for (b) $dV/dt < 0$, and (c) $dV/dt > 0$; add the normal and tangential components by placing the two vectors head to tail)

[3 POINTS – one for each fig]

5. The Pilot's Operating Handbook for a Cessna-172 (see http://www.redskyventures.org/doc/cessna-poh/C172N-1978-POH-S1to7-scanned.pdf) includes the following information:

- Maximum aircraft takeoff weight 2300lb (1043kg)
- Minimum takeoff distance 805ft (245m)
- Short-field takeoff speed 55 knots (28.29m/s)

Use this information to estimate the thrust produced by the aircraft's engine (neglect drag). Express your answer in Newtons.

The acceleration of the aircraft can be calculated using the straight-line motion equations. The position and velocity vectors of the aircraft during the takeoff roll are

$\mathbf{r}(t) = at^2/2\mathbf{i} \qquad \mathbf{v}(t) = at\mathbf{i}$

At the instant of takeoff, we know that the plane reaches its takeoff speed, and its position is located at the end of the runway. Thus, at this instant $\mathbf{v} = V_T\mathbf{i} \quad \mathbf{r} = d\mathbf{i}$. The two vector equations then give

$$d = at^2/2 \quad V_T = at \Rightarrow V_T^2/2d = a$$

Substituting numbers gives a=1.63ms^{-2}

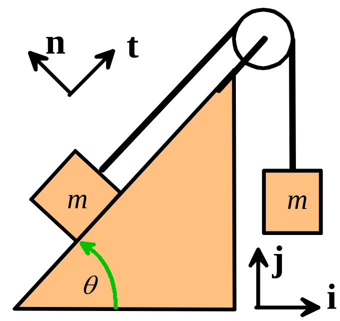
A free body diagram for the aircraft is shown in the figure. Writing down Newton's laws gives

$$F_T\mathbf{i} + (N_1 + N_2 + N_3 - mg)\mathbf{j} = ma\mathbf{i}$$

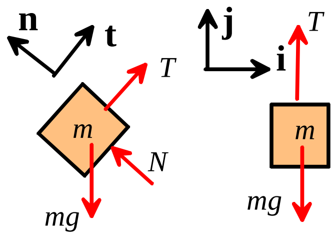
The thrust follows as $F_T = ma$. Substituting numbers gives 1700N.

[4 POINTS]

6. The two blocks shown in the figure have the same mass. The contact between the mass and wedge is frictionless, and the mass of the pulley can be neglected.



6.1 Draw a free body diagram showing the forces acting on each mass



IT IS IMPORTANT TO SHOW THE TWO MASSES ISOLATED FROM THE REST OF THE SYSTEM, OTHERWISE IT IS NOT CLEAR WHAT THE FORCES ARE ACTING ON.

[3 POINTS]

6.2 Write down a relationship between the magnitudes a_1, a_2 of the accelerations of the two masses

Since the length of the cable remains constant, both masses must move at the same speed, so $a_1 = a_2$

[1 POINT]

6.3 Write down a unit vector parallel to the direction of the acceleration of each mass. Use whichever basis you find convenient

For the mass on the wedge, $\mathbf{a} = a_1\mathbf{t} = a_1(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$

For the suspended mass $\mathbf{a} = -a_1\mathbf{j} = -\sin\theta\mathbf{t} - \cos\theta\mathbf{n}$

[2 POINTS]

6.4 Write down Newton's laws of motion for both masses

For the mass on the wedge,

$$m\mathbf{a} = ma_1\mathbf{t} = T\mathbf{t} + N\mathbf{n} - mg(\cos\theta\mathbf{n} + \sin\theta\mathbf{t})$$
$$= ma_1(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) = T(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) - mg\mathbf{j} + N(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})$$

For the suspended mass $m\mathbf{a} = -ma_1\mathbf{j} = (T - mg)\mathbf{j}$

[2 POINTS]

6.5 Finally, determine the tension in the cable, and the acceleration of each mass

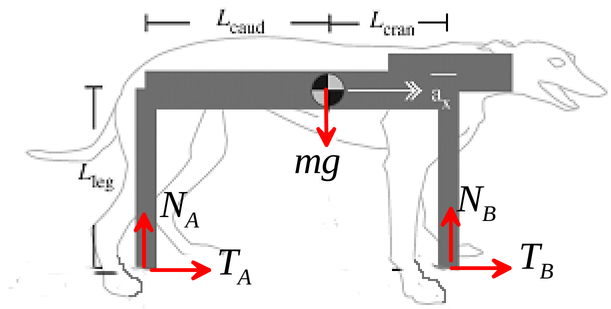
Here's a MAPLE solution to the three scalar equations

```
> restart:
> eq1 := m*a1=T-m*g*sin(theta):
> eq2 := 0=N-m*g*cos(theta):
> eq3 := -m*a1=(T-m*g):
> solve({eq1,eq2,eq3},{a1,T,N});
```

$$\{T = \frac{m\,g}{2} + \frac{1}{2}m\,g\,\sin(\theta),\ a1 = -\frac{1}{2}g\,(-1 + \sin(\theta)),\ N = m\,g\,\cos(\theta)\}$$

[2 POINTS]

7. Experimental measurements suggest that athletes, and running animals, cannot accelerate faster than a critical limit (see, e.g. Williams *et al* 2009). At high speeds, the maximum possible acceleration is limited by the maximum power that the animal's muscles can develop. At low speeds, the maximum possible acceleration is limited by basic physics: if the animal tries to accelerate to fast, it tips over. The goal of this problem is to determine the maximum low-speed acceleration of a quadruped.



7.1 Draw the forces acting on a copy of the idealized stick-animal shown in the figure during its initial acceleration

The forces are shown on the diagram above.

[2 POINTS]

7.2 Assume that the center of mass of the animal remains at a fixed height, so the animal's acceleration vector is $a\mathbf{i}$, where a is to be determined. Use Newton's laws **F**=m**a** and the rotational equation of motion **M=0** to calculate formulas for the reaction forces acting on the animal's feet, in terms of a and the dimensions shown in the figure.

The equations of motion are

$$\mathbf{F} = m\mathbf{a} \Rightarrow (T_A + T_B)\mathbf{i} + (N_A + N_B - mg)\mathbf{j} = ma_x\mathbf{i}$$

$$\mathbf{M} = \mathbf{0} \Rightarrow \left[(T_A + T_B)L_{leg} + (N_B L_{cran} - N_A L_{caud}) \right]\mathbf{k} = \mathbf{0}$$

These three equations can be solved for $N_A, N_B, (T_A + T_B)$. Of course T_A, T_B can't both be determined – only their sum can be found (there are four unknowns and only 3 equations).

```
> restart:
> eq1 := TA+TB=m*a:
> eq2 := NA+NB-m*g=0:
> eq3 := (TA+TB)*Lleg + (NB*Lcran-NA*Lcaud)=0:
> solve({eq1,eq2,eq3},{NA,NB,TA,TB});
```

$$\left\{ TA = -TB + m\,a,\ NB = -\frac{m\,(Lleg\,a - Lcaud\,g)}{Lcran + Lcaud},\ NA = \frac{m\,(Lleg\,a + g\,Lcran)}{Lcran + Lcaud},\ TB = TB \right\}$$

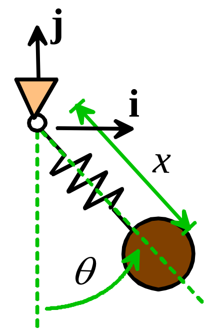
[2 POINTS]

7.3 Hence, find a formula for the maximum possible value of a that can be achieved without the animal's front legs leaving the ground.

$$N_B > 0 \Rightarrow L_{caud}\, g - L_{leg}\, a > 0 \Rightarrow a < gL_{caud} / L_{leg}$$

[1 POINT]

8. The figure shows a mass at the end of a flexible cable, which is idealized as a spring. The motion of the system is characterized by the length of the spring, x, and the angle θ.



8.1 Write down the position vector of the mass in terms of these variables, expressing your answer as components in the basis {**I,j,k**} shown in the figure.

The position vector is

$$\mathbf{r} = x\sin\theta\mathbf{i} - x\cos\theta\mathbf{j}$$

[1 POINT]

8.2 Find formulas for the velocity and acceleration vectors of the mass, in terms of x, θ and their time derivatives. You can use MAPLE to do this if you like – but you will need to be able to do this kind of calculation by hand in exams, so make sure you understand what MAPLE is doing!

Here's the MAPLE solution

```
> with(VectorCalculus):
> r := <x(t)*sin(theta(t)),-x(t)*cos(theta(t))>:
> v := simplify(diff(r,t));
```

$$v := \left(\left(\frac{d}{dt}x(t)\right)\sin(\theta(t)) + x(t)\cos(\theta(t))\left(\frac{d}{dt}\theta(t)\right)\right)\mathbf{e}_{\mathbf{x}} + \left(-\left(\frac{d}{dt}x(t)\right)\cos(\theta(t)) + x(t)\sin(\theta(t))\left(\frac{d}{dt}\theta(t)\right)\right)\mathbf{e}_{\mathbf{y}}$$

```
> a := simplify(diff(v,t));
```

$$a := \left(\left(\frac{d^2}{dt^2}x(t)\right)\sin(\theta(t)) + 2\left(\frac{d}{dt}x(t)\right)\cos(\theta(t))\left(\frac{d}{dt}\theta(t)\right) - x(t)\sin(\theta(t))\left(\frac{d}{dt}\theta(t)\right)^2 + x(t)\cos(\theta(t))\left(\frac{d^2}{dt^2}\theta(t)\right)\right)\mathbf{e}_{\mathbf{x}} + \left(-\left(\frac{d^2}{dt^2}x(t)\right)\cos(\theta(t)) + 2\left(\frac{d}{dt}x(t)\right)\sin(\theta(t))\left(\frac{d}{dt}\theta(t)\right) + x(t)\cos(\theta(t))\left(\frac{d}{dt}\theta(t)\right)^2 + x(t)\sin(\theta(t))\left(\frac{d^2}{dt^2}\theta(t)\right)\right)\mathbf{e}_{\mathbf{y}}$$

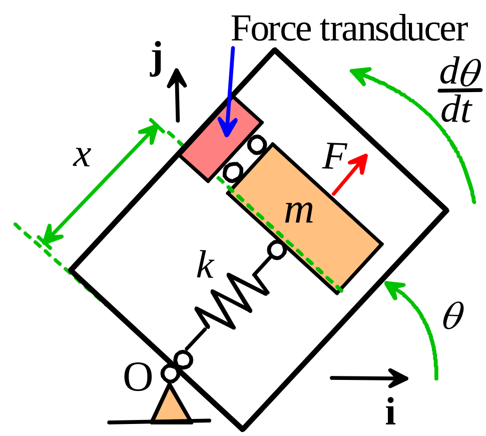
These results can be written in a slightly simpler form as

$$\mathbf{v} = \frac{dx}{dt}(\sin\theta\mathbf{i} - \cos\theta\mathbf{j}) + x\frac{d\theta}{dt}(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

$$\mathbf{a} = \left(\frac{d^2x}{dt^2} - x\left(\frac{d\theta}{dt}\right)^2\right)(\sin\theta\mathbf{i} - \cos\theta\mathbf{j}) + \left(x\frac{d^2\theta}{dt^2} + 2\frac{dx}{dt}\frac{d\theta}{dt}\right)(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

[4 POINTS]

9. The figure shows a design for a MEMS gyroscope, which is used to sense rotations (for more information, see, e.g. http://www.sensorsmag.com/sensors/acceleration-vibration/an-overview-mems-inertial-sensing-technology-970) The device consists of a small mass *m* inside a rigid case. The entire assembly rotates at a **constant** rate $d\theta/dt$ about the pivot shown: the purpose of the sensor is to measure this rotation. The mass is supported by a spring with stiffness *k*, and is subjected to a time-varying force $F(t) = F_0 \sin\omega t$ by an electrostatic actuator. This causes the spring/mass system to vibrate, so that the length of the spring has the form $x(t) = L + X_0 \sin\omega t$. A force transducer then measures the reaction force *N(t)* acting on the mass: this signal is then used to determine the rotation rate $d\theta/dt$. The goal of this problem is to derive the formula relating *N* to $d\theta/dt$



9.1 Write down an expression for the position vector of the mass relative to the pivot at O, in terms of *x* and θ

We can do this by trig, or just write down the answer by adding $\pi/2$ to θ in the solution to problem 8, and noting $\sin(\pi/2+\theta) = \cos\theta$, $\cos(\pi/2+\theta) = -\sin\theta$. This gives $\mathbf{r} = x\cos\theta\mathbf{i} + x\sin\theta\mathbf{j}$

[1 POINT]

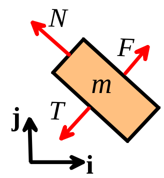
9.2 Hence, calculate formulas for the acceleration vector of the mass, in terms of *x*, θ and their time derivatives. You do not need to use Newton's laws to answer this.

We can use the answer to problem 8, – but setting $d^2\theta/dt^2 = 0$ and adding $\pi/2$ to θ. Or we can just crank through the time derivatives to get

$$\mathbf{a} = \left(\frac{d^2x}{dt^2} - x\left(\frac{d\theta}{dt}\right)^2\right)(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) + \left(2\frac{dx}{dt}\frac{d\theta}{dt}\right)(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})$$

[2 POINTS]

9.3 Draw a free body diagram showing the forces acting on the mass. Gravity can be neglected.



[2 POINTS]

9.4 Hence, write down **F**=*m***a** for the mass.

$$m\left(\frac{d^2x}{dt^2} - x\left(\frac{d\theta}{dt}\right)^2\right)(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) + m\left(2\frac{dx}{dt}\frac{d\theta}{dt}\right)(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})$$
$$= (F - T)(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) + N(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})$$

[2 POINTS]

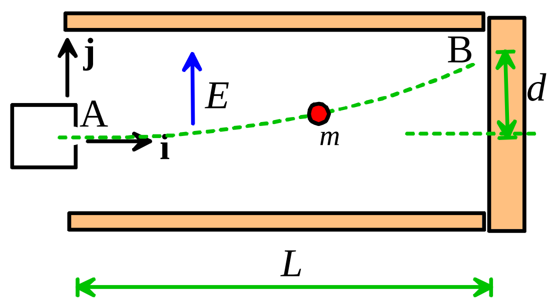
9.5 Finally, show that $d\theta / dt = N / (2mdx / dt)$

The quickest way to do this is to take the dot product of both sides of 9.4 with $(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})$, which gives

$$m\left(2\frac{dx}{dt}\frac{d\theta}{dt}\right) = N$$

[2 POINTS]

10. The figure shows a proposed design for an electrostatic mass spectrometer, which is intended to be used as a carbon-13 urea breath test (used to detect the bacteria that cause stomach ulcers). In this test, the patient swallows urea containing carbon-13. The mass spectrometer is then used to test the patient's breath for the presence of isotope labeled CO2. If the isotope is detected, the urea was split in the patients stomach, indicating that H. pylori bacteria are present. See, e.g. http://www.ncbi.nlm.nih.gov/pmc/articles/PMC1766662/ for more information.



The goal of this problem is to work through a few preliminary design calculations for the mass spectrometer.

- Positively charged CO2 ions are emitted with speed $V = \sqrt{2e\Phi / m}$ at A, where e is the charge of an electron, Φ is the acceleration voltage, and m is the mass of the ion.
- Between A and B, an electric field E exerts a vertical force with magnitude eE (where e is the charge of an electron) on the ions
- The ions strike detectors at B.

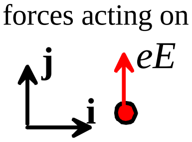
Because the carbon-13 isotope has a different mass to carbon 12, it strikes the detector at a different position.

10.1 The motion of an ion can be described by its (x,y) coordinates. Write down the acceleration vector of an ion in terms of these variables.

This is just the definition of acceleration $\mathbf{a} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}$

[1 POINT]

10.2 Draw a free body diagram showing the forces acting on the ion. Gravity can be neglected.



[1 POINT]

10.3 Write down Newton's law of motion for the ion

$$m\mathbf{a} = m\frac{d^2x}{dt^2}\mathbf{i} + m\frac{d^2y}{dt^2}\mathbf{j} = eE\mathbf{j}$$

[1 POINT]

10.4 Hence, calculate formulas for *x* and *y* as a function of time.

The **i**,**j** components or 10.3 give $\frac{d^2x}{dt^2} = 0 \qquad \frac{d^2y}{dt^2} = \frac{Ee}{m}$

The initial conditions are $x = 0 \quad \frac{dx}{dt} = V \qquad y = \frac{dy}{dt} = 0$

The solution follows (you can use the MAPLE dsolve function if you can't solve these by hand – see below) as $x = Vt \quad y = Eet^2/2m$

```
> with(DETools):
> EOM := {diff(x(t),t$2)=0,diff(y(t),t$2)=E*e/m}:
> IC := {D(x)(0)=V,x(0)=0,D(y)(0)=0,y(0)=0}:
> dsolve(EOM union IC,{x(t),y(t)});
```

$$\{y(t) = \frac{E\,e\,t^2}{2\,m},\ x(t) = V\,t\}$$

[4 POINTS]

10.5 Show that the vertical deflection *h* of an ion when it reaches the detectors can be calculated as

The position vector of the particle can be expressed as

$$Vt\mathbf{i} + \frac{Eet^2}{2}\mathbf{j} = L\mathbf{i} + d\mathbf{j}$$

The **i**, **j** components give two equations that can be solved for the impact time *t* and the deflection *d*, giving

$$t = L/V$$

$$d = \frac{Ee}{2m}\left(\frac{L}{V}\right)^2$$

Substituting for *V* gives

$$d = \frac{Ee}{2m}\left(\frac{L}{\sqrt{2e\Phi/m}}\right)^2 = \frac{EL^2}{4\Phi}$$

This design will not work – the deflection is independent of the mass of the ion.

[2 POINTS]