

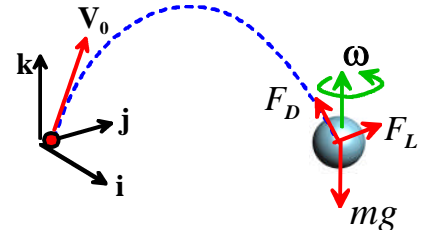


## EN40: Dynamics and Vibrations

### Homework 3: Solving equations of motion for particles Due Friday Feb 19 2010

Division of Engineering  
Brown University

1. A spherical projectile with diameter  $D$  and mass  $m$  is launched from the origin with initial velocity vector  $\mathbf{V}_0 = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}$  and angular velocity (spin)  $\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$ . The projectile is subjected to the force of gravity (acting in the negative  $\mathbf{k}$  direction) together with lift and drag forces



$$\mathbf{F}_L = \rho C_L \frac{\pi D^2}{4} \boldsymbol{\omega} \times \mathbf{v} \quad \mathbf{F}_D = -\frac{1}{2} \rho C_D \frac{\pi D^2}{4} V \mathbf{v}$$

where  $V = \sqrt{v_x^2 + v_y^2 + v_z^2}$  is the magnitude of the particle's velocity relative to the air,  $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$  is the projectile velocity,  $C_L$  and  $C_D$  are the lift and drag coefficients, and  $\rho$  is the air density.

1.1 The motion of the system will be described using the  $(x,y,z)$  coordinates of the particle. Write down the acceleration vector in terms of time derivatives of these variables.

1.2 Write down the vector equation of motion for the particle ( $\mathbf{F}=\mathbf{ma}$ ).

1.3 Show that the equation of motion can be expressed in MATLAB form as

$$\frac{d}{dt} \begin{bmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ -c_1 V v_x + c_2 (\omega_y v_z - \omega_z v_y) \\ -c_1 V v_y + c_2 (\omega_z v_x - \omega_x v_z) \\ -g - c_1 V v_z + c_2 (\omega_x v_y - \omega_y v_x) \end{bmatrix}$$

and give formulas for the constants  $c_1$  and  $c_2$ .

1.4 Modify the MATLAB code discussed in class (or the online notes, or the MATLAB tutorial, or last year's HW3) to calculate and plot the trajectory. Add an 'Event' function to your code that will stop the calculation when the projectile reaches a critical *horizontal* distance  $d$  from the launch point (i.e.  $\sqrt{x^2 + y^2} = d$ ). (THERE IS NO NEED TO SUBMIT A SOLUTION TO THIS PROBLEM)

1.5 Finally, run simulations with parameters representing a baseball ( $m=0.142\text{kg}$ ,  $C_D = 0.3$ ,  $C_L = 0.01$ ,  $D=7.48\text{cm}$ , air density  $1.02 \text{ kg/m}^3$ ) launched at  $34.4 \text{ m/s}$  ( $77 \text{ mph}$ ) at an angle of  $1$  degrees to the horizontal. Run simulations with  $\boldsymbol{\omega} = \mathbf{0}$  and  $\boldsymbol{\omega} = 132\mathbf{k} \text{ rad/sec}$  (the second case corresponds to a 'slider' from a right-handed pitcher). In each case, find the position vector of the ball when it crosses the plate (at a distance  $d=18.44\text{m}$  from the launch point). Hence, calculate the deflection of the ball due to spin. There is no need to hand in MATLAB code or graphs for this problem – just report your values.

2. In this problem, you will write a MATLAB script to predict the orbit of a satellite around the earth. This is the first step in the calculations you will need to do for the lunar impact design project later in the semester. To simplify calculations:

- Neglect the gravitation of the moon and sun
- Assume that the earth is stationary

2.1 The position of the satellite will be described using its  $(x,y,z)$  coordinates (the design project handout describes how the  $\mathbf{i},\mathbf{j},\mathbf{k}$  directions are chosen, but this is not important here). Write down the velocity and acceleration of the satellite in terms of these variables

2.2 Draw a free body diagram showing the forces acting on the satellite.

2.3 Write down Newton's law of motion  $\mathbf{F}=\mathbf{ma}$  for the satellite.

2.4 Show that the equations of motion can be expressed in MATLAB form as

$$\frac{d}{dt} \begin{bmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ -GMx/R^3 \\ -GM_y/R^3 \\ -GM_z/R^3 \end{bmatrix}$$

where  $G$  is the gravitational constant,  $M$  is the mass of the earth, and  $R = \sqrt{x^2 + y^2 + z^2}$  is the distance of the satellite from the earth's center.

2.5 write a MATLAB script that will integrate the equations of motion. Be sure to use the 'odeset' command to control the accuracy of the computation, as discussed in class. (THERE IS NO NEED TO SUBMIT A SOLUTION TO THIS PROBLEM).

2.6 Test your code by running it with the following parameters:

- $\mu = GM = 3.986012 \times 10^5 \text{ km}^3\text{s}^{-1}$
- Initial position:  $\mathbf{r} = -19063\mathbf{i} + 18504\mathbf{j} + 760\mathbf{k}$  km,
- Initial velocity:  $\mathbf{v} = -1.6128\mathbf{i} - 1.5314\mathbf{j} - 3.1703\mathbf{k}$  km/sec

(this is a typical orbit for a GPS satellite). Hand in a plot showing the satellite orbit.

2.7 Add an 'Event' function to your code that will determine the times that the satellite crosses the equatorial plane  $z=0$ . (See the MATLAB tutorial for a description of how to use the 'event' function to identify more than one event) Hence, determine the time for one complete orbit.

3. A ‘Penning Trap’ is a device that uses specially shaped electrostatic and electromagnetic fields to trap charged particles (or ‘ions’) (see, e.g. [Major et al \(2005\)](#) for more information). It has a number of applications: for example, as a mass spectrometer (used in chemical and biochemical analysis); or as a component in experimental quantum computers. The ions inside a Penning trap are continuously moving, and follow very complicated orbits. The goal of this problem is to analyze the motion of a single charged particle inside a Penning trap.

3.1 The motion of the ion will be described by the components of its position vector  $(x,y,z)$ . Write down the velocity and acceleration of the ion in terms of these variables.

3.2 The ion has mass  $m$ , and is subjected to a force

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

where  $\mathbf{r}$  is the ion’s (instantaneous) position vector,  $\mathbf{v}$  is its velocity vector,  $Q$  is its charge, and

$$\mathbf{E} = \frac{E_0}{d}(x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}) \quad \mathbf{B} = B_0\mathbf{k}$$

are the electric and magnetic field vectors. Here,  $E_0, d, B_0$  are constants that specify the magnitude and geometry of the electric and magnetic fields. Use Newton’s law to show that the components of the position vector of the ion satisfies the following equations of motion

$$\frac{d^2x}{dt^2} - \omega_c v_y - \frac{\omega_z^2}{2}x = 0 \quad \frac{d^2y}{dt^2} + \omega_c v_x - \frac{\omega_z^2}{2}y = 0 \quad \frac{d^2z}{dt^2} + \omega_z^2 z = 0$$

where

$$\omega_c = \frac{QB_0}{m} \quad \omega_z = \sqrt{\frac{2QE_0}{dm}}$$

are two parameters (known as the ‘cyclotron frequency’ and ‘axial frequency’ of the system).

3.3 Re-write the equations of motion in problem 2.2 as 6 first-order differential equations that can be integrated using MATLAB.

3.4 Write a MATLAB script that will solve the equations of motion to determine  $x, y, z, v_x, v_y, v_z$  as a function of time. You need not submit a solution to this problem.

3.5 An ion will be trapped if the frequencies  $\omega_c, \omega_z$  satisfy the condition  $\omega_c^2 - 2\omega_z^2 > 0$ . Check this prediction by running two simulations with the following parameters:

(a)  $x = 0.1 \quad y = z = 0 \quad v_x = v_y = v_z = 0 \quad \omega_c = 1.415 \quad \omega_z = 1$

(b)  $x = 0.1 \quad y = z = 0 \quad v_x = v_y = v_z = 0 \quad \omega_c = 1.413 \quad \omega_z = 1$

Run both tests for a time interval of 300 units. Hand in a graph showing the predicted trajectory for each case.

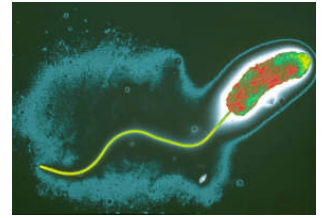
When used as a spectrometer, the ion trajectories are used (indirectly) to determine values for  $\omega_c, \omega_z$ , and hence to find the mass of the particle. To test the influence of  $\omega_c, \omega_z$  on the trajectories, run simulations with

(c)  $x = 0.1 \quad y = z = 0 \quad v_x = v_y = 0 \quad v_z = 0.1 \quad \omega_c = 1.5 \quad \omega_z = 1$

(d)  $x = 0.1 \quad y = z = 0 \quad v_x = v_y = 0 \quad v_z = 0.1 \quad \omega_c = 2.125 \quad \omega_z = 1.5$

Hand in a two graphs showing (in 3D) the trajectory for each case.

4. **OPTIONAL PROBLEM FOR EXTRA CREDIT.** There is currently great interest in analyzing the propulsion mechanisms and motion of bacteria. This is partly because bacteria can be genetically engineered to perform useful tasks (for example, bacteria have been used as microfluidic pumps), and partly because there are too many physicists with not enough to do.



Many bacteria propel themselves by means of a rotating tail (a flagellum) as shown in the figure (from Nature 406, 469-470 3 August 2000) doi:10.1038/35020178

In this problem, you will set up and analyze a simple model that is used to describe the motion of a bacterium swimming through a viscous fluid.

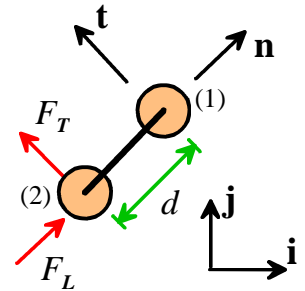
The body of the bacterium is represented by two beads with mass  $m$ , connected by a rigid link with length  $d$ , as shown in the figure. Each bead is subjected to a viscous drag force

$$\mathbf{F}_D = -\eta \mathbf{v}$$

where  $\mathbf{v}$  is the velocity vector of the bead, and  $\eta$  is the viscosity of the fluid. The bacterium is propelled by its rotating flagellum, which is attached to the rearmost bead. The flagellum exerts a constant force  $F_L$  parallel to the body of the bacterium, together with a small fluctuating transverse force  $F_T(t) = F_T \sin \omega t$ . These forces can be expressed as vectors

$$\mathbf{F}_L = F_L \mathbf{n} \quad \mathbf{F}_T = F_T \sin \omega t \mathbf{t}$$

where  $\mathbf{n}$  and  $\mathbf{t}$  are unit vectors parallel and perpendicular to the rigid link, as shown in the figure. For simplicity, we will analyze motion only in two dimensions.



4.1 The motion of the bacterium will be described using the position vectors of the two beads

$\mathbf{r}_1 = x_1 \mathbf{i} + y_1 \mathbf{j}$   $\mathbf{r}_2 = x_2 \mathbf{i} + y_2 \mathbf{j}$ . Write down the velocity and acceleration vector of each bead in terms of time derivatives of  $(x_1, y_1), (x_2, y_2)$ .

4.2 Find formulas for the components of the unit vectors  $\mathbf{n}$  and  $\mathbf{t}$  in terms of  $x_1, y_1, x_2, y_2$  and  $d$

4.3 The length of the rigid link remains constant. This requires that  $(\mathbf{r}_1 - \mathbf{r}_2) \cdot (\mathbf{r}_1 - \mathbf{r}_2) = d^2$ . By differentiating this expression with respect to time, show that the accelerations of the two beads must be related by the following constraint

$$(\mathbf{a}_1 - \mathbf{a}_2) \cdot (\mathbf{r}_2 - \mathbf{r}_1) = (\mathbf{v}_1 - \mathbf{v}_2) \cdot (\mathbf{v}_1 - \mathbf{v}_2)$$

4.4 Draw two free body diagrams showing the forces acting on each bead. Note that the rigid link exerts equal and opposite forces on the two particles. The forces have unknown magnitude  $R$ , but must be parallel to the link.

4.5 Write down Newton's laws of motion  $\mathbf{F} = m\mathbf{a}$  for each bead.

4.6 Hence, show that the equations of motion for the two beads, together with the constraint equation in problem (3), can be expressed in MATLAB form as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m & 0 & 0 & 0 & (x_2 - x_1)/d & \dot{x}_1 \\ 0 & 0 & 0 & 0 & 0 & m & 0 & 0 & (y_2 - y_1)/d & \dot{y}_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & m & 0 & (x_1 - x_2)/d & \dot{x}_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & m & (y_1 - y_2)/d & \dot{y}_2 \\ 0 & 0 & 0 & 0 & (x_2 - x_1) & (y_2 - y_1) & (x_1 - x_2) & (y_1 - y_2) & 0 & R \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{x}_2 \\ \dot{y}_2 \\ \dot{x}_1 \\ \dot{y}_1 \\ \dot{x}_2 \\ \dot{y}_2 \\ R \end{bmatrix} = \begin{bmatrix} v_{x1} \\ v_{y1} \\ v_{x2} \\ v_{y2} \\ -\eta v_{x1} \\ -\eta v_{y1} \\ F_L n_x + F_T t_x - \eta v_{x2} \\ F_L n_y + F_T t_y - \eta v_{y2} \\ (v_{x1} - v_{x2})^2 + (v_{y1} - v_{y2})^2 \end{bmatrix}$$

where  $R$  is the unknown reaction force in the link, the dots over the variables represent time derivatives, and  $n_x = (x_1 - x_2)/d$ ,  $n_y = (y_1 - y_2)/d$ ,  $t_x = -(y_1 - y_2)/d$ ,  $t_y = (x_1 - x_2)/d$

4.7 Write a MATLAB script that will calculate and plot the trajectory of the bacterium (you only need to plot the trajectory of one of the two beads). There is no need to submit a solution to this problem.

4.8 Test your code by running it with  $m=0.1$  and the following parameters:

- $\eta = 10$ ,  $\omega = 0.1$ ,  $F_T = 0.01$ ,  $F_L = 1$ ,  $d = 0.01$ , initial position  $x_1 = y_1 = 0$ ,  $x_2 = -d$ ,  $y_2 = 0$  and zero initial velocity
- $\eta = 10$ ,  $\omega = 0.1$ ,  $F_T = 0.01$ ,  $F_L = 1$ ,  $d = 0.001$ , initial position  $x_1 = y_1 = 0$ ,  $x_2 = -d$ ,  $y_2 = 0$  and zero initial velocity.

Hand in plots showing the predicted trajectory for each case.