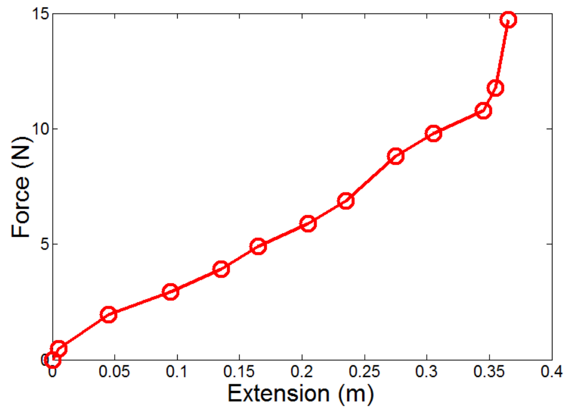
Division of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 4: Work, Energy and Momentum
Solutions. MAX SCORE 70 POINTS

1. Do a simple experiment to measure the force-v-extension curve for a rubber band. Describe your experimental method briefly, and use MATLAB to plot a graph showing the force-v-extension curve. Calculate the maximum energy that can be stored in the elastic band before it breaks. (you can use the MATLAB 'trapz' function to integrate your force-extension curve. Be careful with units!).



My data (obtained using the masses and spring scales provided in 096) is shown in the figure.

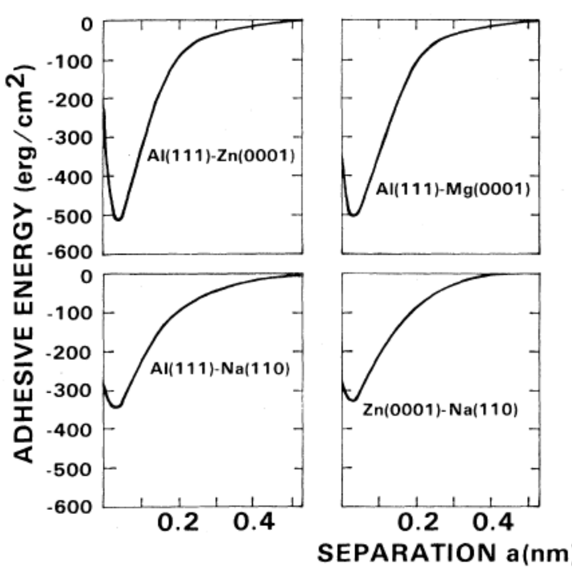
The energy is the area under the curve, 2.1J

[5 points]

OPTIONAL: Not for credit. Hence, estimate the maximum distance you can fire the elastic band when standing on level ground. Verify your prediction experimentally.

2. The work per unit area required to separate two atomic planes in a crystal by a distance x can be approximated by the 'Universal Binding Energy Relation' (Rose, Ferrante and Smith, Phys Rev Lett, **47** (1981)), given by

$$E(x) = E_0 - E_0\left(1+\frac{x}{d}\right)\exp\left(-\frac{x}{d}\right)$$



where E_0 is the total work of separation for the interface, and d is a characteristic length. (Note that in this expression the energy is zero at $x=0$, and increases to a value E_0 for large x. In the data shown in the figure, the energy is zero for large x, and the minimum energy corresponds to $x=0$.)

2.1 Find an expression for the force of attraction per unit area between the two crystallographic planes

$$\mathbf{F}(x) = -\frac{dE}{dx}\mathbf{i} = \frac{E_0}{d^2}x\exp\left(-\frac{x}{d}\right)\mathbf{i}$$

[1 POINT]

2.2 Find an expression for the maximum force of attraction per unit area, in terms of E_0 and d.

The max occurs where $dF/dx = 0$, i.e.

$$\frac{E_0}{d^3}(d-x)\exp\left(-\frac{x}{d}\right)=0$$

or $x=d$. The corresponding maximum force is $E_0 \exp(-1)/d$

[1 POINT]

2.3 Hence, estimate the strength of interplanar bond between Aluminum and Zinc, using data from Rose, Ferrante and Smith (1981). Express your answer in $\mathrm{MNm^{-2}}$

We need to estimate E_0 and d from the data. The value of E_0 can be determined from the minimum energy, which is approximately 510 $\mathrm{ergs/cm^2}$

The value of d can be estimated by choosing a second point on the curve – e.g. by noting that $E = 460\ \mathrm{ergs/cm^2}$ at $x=0.2nm$. We can substitute these values into the formula and solve for d

$$460 = 510 - 510\left(1+\frac{2}{d}\right)\exp\left(-\frac{2}{d}\right)$$

```
> 460=510*(1-(1+0.2/d)*exp(-0.2/d));
```

$$460 = 510 - 510\left(1+\frac{0.2}{d}\right)\mathbf{e}^{\left(-\frac{0.2}{d}\right)}$$

```
> solve(%,d);
```

$$-0.2077798255,\ 0.05109081968$$

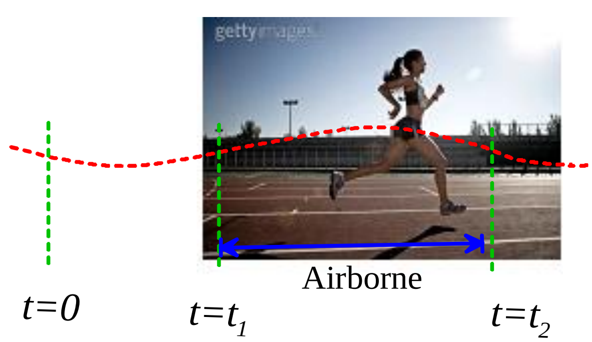
This gives $d=0.051$nm. Finally, note that 1 erg is 10^{-7}Joules, and 1nm is 10^{-9}m so

$F_{max} = E_0 \exp(-1)/d = 3.6\times10^5\ N/cm^2$ or 3700 $\mathrm{MNm^{-2}}$

There are many other ways to estimate d – the values may vary depending on the method chosen. Any reasonable approach that gives the solution to within an order of magnitude should get credit, provided that the procedure followed is explained clearly.

[3 POINTS]

3. The goal of this problem is to estimate the power expended by a running athlete. The figure shows the trajectory of the center of mass of a runner. It consists of two portions: from $0 < t < t_1$ the runner's foot is in contact with the ground; while between $t_1 < t < t_2$ the runner is airborne. For simplicity, assume that the horizontal component of the runner's velocity V is constant.



3.1 By considering the motion of the runner while she is airborne, calculate the vertical component of her velocity at the instant when she leaves the ground, in terms of the gravitational acceleration g and t_1 and t_2 (neglect air resistance) and the runner's mass m. Hence, write down the runner's kinetic energy at this instant.

We can do this using the trajectory equations.

(i) At the launch point, the runner has unknown vertical speed V_{y0}

(ii) The runner reaches the top of her trajectory in time $(t_2 - t_1)/2$ (the trajectory is symmetric)

(iii) The trajectory equations give $v_y = V_{y0} - gt$, so $V_{y0} = g(t_2 - t_1)/2$

(iv) The kinetic energy follows as $KE = m(v_x^2 + v_y^2)/2 = mV^2/2 + mg^2(t_2 - t_1)^2/8$

[5 POINTS]

3.2 Write down the runner's kinetic energy at the instant that her center of mass reaches its minimum height, in terms of her mass m and V.

At this position her vertical speed is zero, so $KE = mV^2/2$

[**1 POINT]**

3.3 Assume that the additional kinetic energy required for the athlete to become airborne is dissipated as heat during the subsequent impact of her foot with the ground. Hence, calculate a formula for the power lost to impacts, in terms of g, t_1 and t_2 and m.

The power is the change in KE per step, divided by the time for a step, i.e.

$$P = mg^2(t_2 - t_1)^2 / 8t_2$$

[1 POINT]

3.4 Write down a formula for the power expended by the runner to overcome air resistance.

The drag force is $\frac{1}{2}\rho C_D A V^2$, where ρ is the air density, A is the projected frontal area of the runner, C_D is the drag coefficient, and V is her horizontal speed.

The power follows as $\frac{1}{2}\rho C_D A V^3$

[1 POINT]

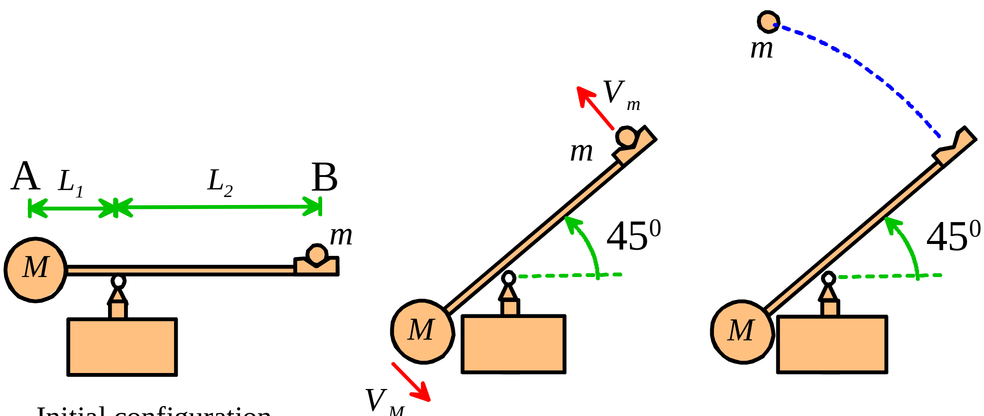
3.5 In a representative stride, a runner has a foot in contact with the ground for 0.1sec and is airborne for 0.2 sec. (see, e.g. Weyand *et al* 'Ambulatory estimates of maximal aerobic power from foot-ground contact times and heart rates in running humans,' *J. Appl. Physiol*, **91**, 451-458 2001). Taking the drag coefficient $C_D = 0.3$, and air density 1.02 kg/m^3, estimate the total energy that you would expend if you were to complete a 5k in 20 mins.

(i) The runner's (horizontal) speed is 4.166 m/s
(ii) Frontal area is about 1 m^2
(iii) Power expended in impacts is (for a 70kg runner) is 112Watts
(iv) Power expended in air resistance 11W (very small!)

The total power is therefore 123W, and the time is 20mins. This gives 148 kJ.

For comparison, a Snickers bar has 271 calories. A food calorie is actually a kilocalorie or 4187 J. So, the body gets ~1.1 MJ from a snickers bar. The mechanical efficiency of human body is around 15 - 20%, so the energy available for running from a sinckers bar is 165 kJ - 220 kJ. According to the calculation, it takes 148 kJ to run 5k in 20 min, which is equivalent to about one snickers bar

[5 POINTS]



4. A proposed design for a catapult that launches a projectile with mass *m* is shown in the figure. The mass of the arm AB can be neglected, and the masses *m* and *M* can be idealized as particles.

4.1 Take the potential energy of the initial configuration to be zero. Find an expression for the potential energy of the system at the instant of launch, in terms of *m*, *M* , L_1, L_2 and the gravitational acceleration *g*.

$$V = \sum m_i g h_i = (mL_2 - ML_1) g \sin(45) = (mL_2 - ML_1) \frac{g}{\sqrt{2}}$$

(2 POINTS)

4.2 Write down the total kinetic energy of the system at the instant just prior to launch, in terms of *m*, *M*, V_m and V_M

$$T = \left(mV_m^2 + MV_M^2\right) / 2$$

(2 POINTS)

4.3 Write down equations that relate the angular velocity of the arm AB to the magnitude of the velocities V_m and V_M at the instant just prior to launch.

The two particles move around circular arcs with radii L_1 and L_2. Suppose the two masses both move at constant speed. Mass *M* travels a distance $2\pi L_1$ in a complete revolution of the arm, while mass *m* travels a distance $2\pi L_2$. Since they would take the same time to complete one revolution, the ratio of the two velocities must satisfy V_M

$$V_M = \frac{L_1}{L_2} V_m$$

(2 POINTS)

4.4 Show that the projectile velocity at the instant of launch satisfies the equation

$$V_m^2 = \frac{\sqrt{2}g(ML_1 - mL_2)}{m + M\dfrac{L_1^2}{L_2^2}}$$

Energy is conserved so *T*+*V*=0 (initial PE and KE are zero). Thus

$$T = \left(mV_m^2 + MV_M^2\right)/2 + \left(mL_2 - ML_1\right)\frac{g}{\sqrt{2}} = 0$$

From 4.3 we see that $V_M = \dfrac{L_1^2}{L_2^2} V_m$ and so

$$T = V_m^2\left(m + M\frac{L_1^2}{L_2^2}\right) = -\left(mL_2 - ML_1\right)\frac{2g}{\sqrt{2}}$$

Which clearly yields the solution given.

(4 POINTS)

4.5 The solution to 4.4 can be written $V_m^2 = \dfrac{\sqrt{2}gL_1(1-\mu\rho)\rho^2}{1+\mu\rho^2}$ where $\mu = m/M$ $\rho = L_2/L_1$. Show that the launch velocity is maximized by the value of $\rho = L_2/L_1$ satisfying the equation

$$\mu^2\rho^3 + 3\mu\rho - 2 = 0$$

We need to maximize V_m with respect to ρ. Differentiate:

$$\frac{d}{d\rho}\frac{(1-\mu\rho)\rho^2}{1+\mu\rho^2} = \frac{2\rho-3\mu\rho^2}{(1+\mu\rho^2)} - \frac{(1-\mu\rho)\rho^2}{(1+\mu\rho^2)^2}2\mu\rho = 0$$

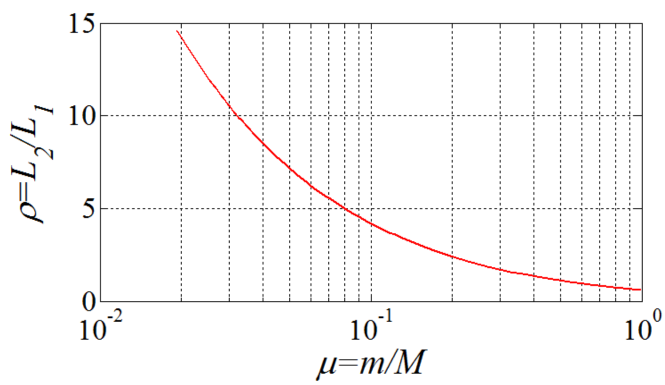
$$\Rightarrow \rho\left(2-3\mu\rho\right)(1+\mu\rho^2) - (1-\mu\rho)\rho^2 2\mu\rho = 0$$

$$\Rightarrow \rho\left(2-3\mu\rho-\mu^2\rho^3\right) = 0$$

$\rho = 0$ is not a useful solution, so the max of interest follows from the equation given.

4.6 Plot a graph showing the optimal value of ρ as a function of μ for some sensible range. (Hint: it is easier to find the value of μ for a given value of ρ than the reverse).

The figure is shown below, together with the MATLAB code that generated the plo



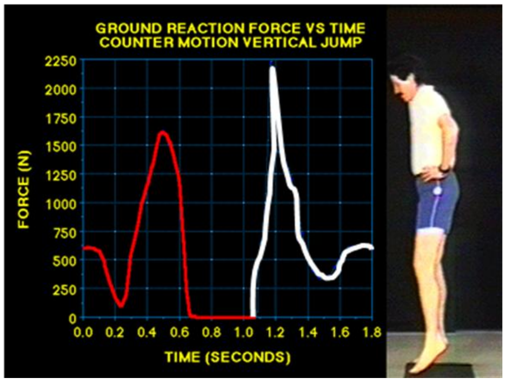

```
function plotrho

for i = 1:100
    rho(i) = 0.6 + 14*(i-1)/99;
    mu(i) = (-3*rho(i) + rho(i)*sqrt(9+8*rho(i)))/(2*rho(i)^3);
end

plot(mu,rho)
end
```

[3 POINTS]

5. The figure shows experimental data for the reaction forces acting on a human test subject during a vertical jump (the red curve corresponds to the jump; the white is the force during the subsequent landing. The person leaves the ground at the point where the red curve drops to zero).



5.1 Assume that at time $t=0$ the subject is just standing on the pressure mat. Use the experimental data to determine the test subject's mass.

The reaction force is 625N. His mass is the force/g, or 63.7kg.

[1 POINT]

5.2 By dividing the force-time curve into a series of straight line segments, calculate (a) the impulse exerted on the person during takeoff by the reaction force at the ground; (b) the impulse exerted on the person by gravity during the period before takeoff, and (c) the total impulse exerted on the person during the period before takeoff.

(a) The table below gives a piecewise-linear interpolation of the test data

Time (s)	Force (N)
0	625
0.1	625
0.21	125
0.5	1625
0.6	1250
0.65	0

The data can be integrated using the MATLAB trapz() command (see below), giving I=523.5 Ns.

```
function compute_impulse
t = [0,0.1,0.21,0.5,0.6,0.65];
f = [625,625,125,1625,1250,0];
imp = trapz(t,f)
end
```

Any sensible and clearly presented method for integrating the impulse should get credit. The actual value is rather sensitive to how many points are chosen and how accurately they are measured, so solutions may vary considerably.

[3 POINTS]

(b) The impulse due to gravity is simply the product of the persons weight and the time of launch, or 625*0.65=406.25Ns.

[1 POINT]

(c) The total impulse is the difference between the two, or 117Ns.

[1 POINT]

5.3. Using the results of 7.1 and 7.2, calculate the vertical velocity of the test subject at the instant of take-off

The net vertical impulse is the change of linear momentum of the test subject, or $I=mv$. His vertical velocity is therefore 1.84m/s.

[1 POINT]

5.4 Hence, determine the height of his jump.

We can do this using energy conservation. The jumper and the earth together are a conservative system. We can assume the earth is stationary. The total KE at the instant of launch is $mv^2/2$. At the top of the trajectory, the person's KE is zero, and PE is mgh. Energy conservation gives

$h = v^2/2g = 0.172m$

[2 POINTS]

5.5 Check your answer using the time that the subject is airborne.

(i) The straight line motion formulas relate the jumper's height h and speed V to the take-off speed V_0 and time t as $h = V_0 t - gt^2/2 \quad V = V_0 - gt$

(ii) Note that the jumper takes the same amount of time to reach the top of the trajectory as he takes return to the ground. If the jump takes a total time T, he reaches the top of the trajectory at time $T/2$.

(iii) Note that at the highest point of the trajectory, the vertical speed is zero

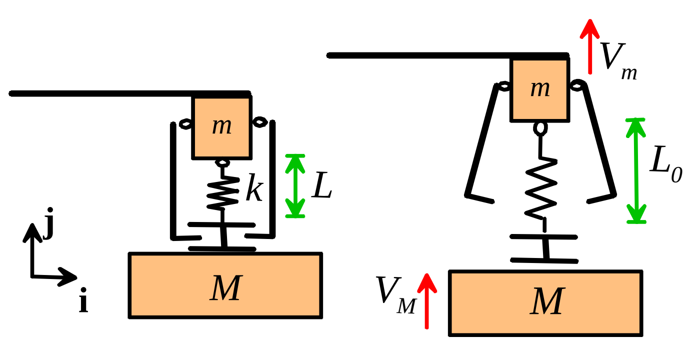
(iv) The previous two observations show that $V_0 = gT/2$. The jump height follows as

$$h = g(T/2)^2 - g(T/2)^2/2 = g(T/2)^2/2$$

(v) From the graph, the total time that the jumper is airborne is about 0.4sec. Substituting numbers gives 0.196m. This is fairly close to the previous estimate. The difference between the two results is most likely because the jumper retracts his feet before landing.

[4 POINTS]

6. An 'Impulse Hammer' is a device that is used to subject a test specimen or structure to a known impulse. The motion of the test structure caused by the impulse can then be used to deduce properties or parameters of interest. For example, impulse response measurements have a number of biomedical applications – see, e.g. Holi and Radhakrishnan, 'In Vivo assessment of osteoporosis in women by impulse response,' TENCON, **4**, 1395, (2003).



The figure shows a proposed design for an impulse hammer. It consists of a mass *m* attached to a spring with unstretched length L_0 and stiffness *k*. Before operation, the spring is pre-compressed to a length *L*. The base is then placed in contact with the specimen, which has mass *M*. The clamps are then released, allowing the spring to extend. The spring exerts equal and opposite impulses on the specimen and the test mass *m*.

6.1 Write down the total linear momentum of the system before the catch is released.

The linear momentum is zero

[1 POINT]

6.2 Write down the total potential and kinetic energy of the system before the catch is released. Gravity may be neglected.

The kinetic energy is zero, the potential energy is the energy stored in the spring: $k(L_0-L)^2/2$

[1 POINT]

6.3 Let V_m, V_M denote the velocities of the two masses after the spring has fully extended. Write down the total linear momentum of the system at this time.

The linear momentum is $\mathbf{p}=\left(MV_M+mV_m\right)\mathbf{j}$

[1 POINT]

6.4 Write down the total potential and kinetic energy of the system at the instant that the spring has fully extended.

The potential energy is zero, and the kinetic energy is $T=MV_M^2/2+mV_m^2/2$

[1 POINT]

6.5 Hence, calculate formulas for V_m, V_M in terms of *m*, *M*, *k*, *L* and L_0.

Energy conservation and momentum conservation give

$$MV_M^2/2+mV_m^2/2=k(L-L_0)^2/2$$

$$\left(MV_M+mV_m\right)=0$$

These equations can be solved for V_m, V_M

```
> restart:
> eq1 := M*VM^2 + m*Vm^2=k*(L0-L)^2:
> eq2 := M*VM+m*Vm=0:
> solve({eq1,eq2},{VM,Vm}):
> convert(%,radical);
```

$$\left\{Vm=\sqrt{\frac{M\,k}{m^2+m\,M}}\,(L0-L),\ VM=-\frac{m\sqrt{\frac{M\,k}{m^2+m\,M}}\,(L0-L)}{M}\right\}$$

or

$$V_m = (L_0 - L)\sqrt{\frac{Mk}{m(M+m)}}, V_M = -(L_0 - L)\sqrt{\frac{mk}{M(M+m)}}$$

[3 POINTS]

6.6 Finally, find an expression for the impulse exerted on the specimen.

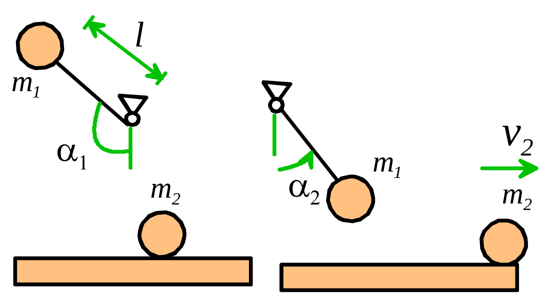
The impulse can be calculated using the impulse-force equation (either the specimen or the mass *m* can be used for this purpose). For mass *M*

$$\Im = M\mathbf{v}_m = -(L_0 - L)\sqrt{\frac{Mmk}{(M+m)}}\mathbf{j}$$

[2 POINTS]

7. The figure shows an experimental apparatus for measuring the restitution coefficient of, e.g. a golf-ball or a bowling ball. It uses the following procedure

- A pendulum (a golf-club head, e.g.) is swung to a known initial angle α_1 and then dropped from rest so as to strike the ball
- The angle of follow-through α_2 of the pendulum is recorded



Your goal is to derive a formula that can be used to determine the restitution coefficient *e* from the measured data.

7.1 Using energy conservation, derive an expression for the speed V of the mass on the end of the pendulum just before it strikes the ball, in terms of α_1, *l* and the gravitational acceleration.

The potential energy of the first mass just before it is dropped is $-m_1 gL\cos\alpha$, and its kinetic energy is zero.

At the instant just before impact its potential energy is $-m_1 gL$, and the kinetic energy is $m_1 V^2/2$

Energy is conserved, so

$$-m_1 gL\cos\alpha_1 = -m_1 gL + m_1 V^2/2$$
$$\Rightarrow V = \sqrt{2gL(1-\cos\alpha_1)}$$

[2 POINTS]

7.2 Similarly, derive an expression for the speed v_1 of the mass on the end of the pendulum just after it strikes the ball, in terms of α_2, *l* and the gravitational acceleration.

The same calculation gives

$$-m_1 gL\cos\alpha_2 = -m_1 gL + m_1 v_1^2/2$$
$$\Rightarrow v_1 = \sqrt{2gL(1-\cos\alpha_2)}$$

[1 POINT]

7.3 Use momentum conservation to calculate an expression for the velocity v_2 of the ball just after it is struck, in terms of V and v_1, and any other necessary parameters.

Momentum is conserved, so that

$$m_1 V = m_1 v_1 + m_2 v_2$$
$$\Rightarrow v_2 = \frac{m_1}{m_2}(V - v_1) = \frac{m_1}{m_2}\sqrt{2gL}\left(\sqrt{1-\cos\alpha_1} - \sqrt{1-\cos\alpha_2}\right)$$

[2 POINTS]

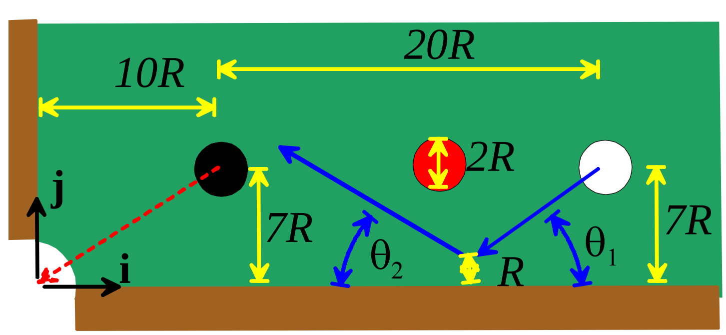
7.4 Hence, deduce a formula for the coefficient of restitution, in terms of α_1, α_2, l, and g, and any other necessary parameters.

The restitution coefficient is the ratio of relative velocities after impact to that before impact, i.e.

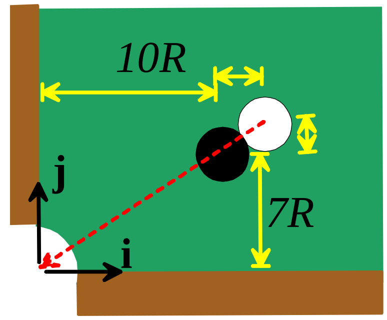
$$\frac{v_2 - v_1}{V} = \frac{\frac{m_1}{m_2}(V - v_1) - v_1}{V}$$
$$= \frac{m_1}{m_2} - (1 + \frac{m_1}{m_2})\frac{v_1}{V} = \frac{m_1}{m_2} - (1 + \frac{m_1}{m_2})\frac{v_1}{V}$$
$$= \frac{m_1}{m_2} - (1 + \frac{m_1}{m_2})\sqrt{\frac{1-\cos\alpha_2}{1-\cos\alpha_1}}$$

[2 POINTS]

8. **Virtual Snooker**. The figure shows three balls on a snooker table. All balls have radius R. The impact of the cue ball (white) with the cushion has a restitution coefficient of 0.8; the impact of two balls has a restitution coefficient of 0.9. Your goal is to calculate an initial velocity for the (white) cue ball that will pot the black ball. Note that you are not allowed to hit the red ball, and so can't make a direct shot – you will have to play the shot off the cushion.



8.1 Write down the coordinates of the cue ball at the instant that it strikes the black ball, assuming that it is on a trajectory to knock the black ball directly into the pocket.



The direction of motion of the black ball after impact is parallel to the line connecting the centers of the two balls at impact, as shown in the figure. Straightforward geometry gives the position vector at impact as

$$\mathbf{r} = 10\left(1+\frac{2}{\sqrt{149}}\right)\mathbf{i} + 7\left(1+\frac{2}{\sqrt{149}}\right)\mathbf{j}$$

[2 POINTS]

8.2 Consider the rebound of the cue ball off the cushion, as shown in the figure. Show that the angle of incidence is related to the angle of rebound by $\tan\theta_2 = e\tan\theta_1$.

Let $\mathbf{v} = v_{x0}\mathbf{i} + v_{y0}\mathbf{j}$ denote the velocity just before impact. Clearly $\tan\theta_1 = v_{y0} / v_{x0}$

After impact, the restitution coefficient formula for 3D impacts gives

$$\mathbf{v}_{A1} - \mathbf{v}_{B1} = \mathbf{v}_{A0} - \mathbf{v}_{B0} - (1+e)[(\mathbf{v}_{A0} - \mathbf{v}_{B0})\cdot\mathbf{n}]\mathbf{n}$$

For this case, the object 'B' is the cushion, which has zero velocity both before and after the impact, and **n** is a unit vector normal to the cushion, i.e. **n=j**. Substituting these into the general formula, and writing $\mathbf{v}_{A0} = v_{x0}\mathbf{i} + v_{y0}\mathbf{j}$, $\mathbf{v}_{A1} = v_{x1}\mathbf{i} + v_{y1}\mathbf{j}$ we see that the velocity components after impact are $\mathbf{v} = v_{x1}\mathbf{i} + v_{y1}\mathbf{j} = v_{x0}\mathbf{i} - ev_{y0}\mathbf{j}$, and furthermore $\tan\theta_2 = -v_{y1} / v_{x1}$. Therefore

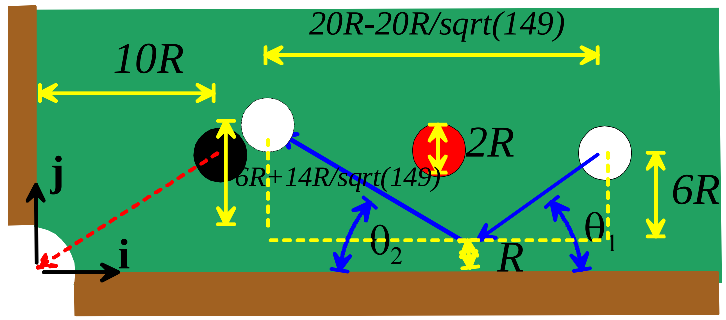
$$\tan\theta_2 = ev_{y0} / v_{x0} = e\tan\theta_1$$

as required.

[2 POINTS]

8.3 Hence, calculate the initial direction of motion of the cue ball in order to make the shot.

Note that the total horizontal distance between the cue ball at its initial position and just before impact is $20R - 2\times\frac{10}{\sqrt{10^2+7^2}} = 20 - \frac{20}{\sqrt{149}}$. We also know the two vertical distances on the dashed yellow triangles shown in the figure below.



We can therefore use trig on the two triangles shown with dashed yellow lines to write down an equation for the total horizontal distance, as follows

$$R(20-20/\sqrt{149}) = \frac{6R}{\tan\theta_1} + \frac{R(6+14/\sqrt{149})}{\tan\theta_2}$$

Finally, we wish to calculate θ_1 so we can express θ_2 in terms of θ_1 using the solution to the preceding part of the problem

$$R(20-20/\sqrt{149}) = \frac{6R}{\tan\theta_1} + \frac{R(6+14/\sqrt{149})}{\tan\theta_2} = \frac{6R}{\tan\theta_1} + \frac{R(6+14/\sqrt{149})}{e\tan\theta_1}$$

$$\Rightarrow \tan\theta_1 = \frac{(6+(6+14/\sqrt{149})/e)}{(20-20/\sqrt{89})} = 0.81331$$

[3 POINTS]

8.4 To check your calculation click here to download a MATLAB simulation of the shot. Save the code in a file called *potit.m* and then run it with

```
>>potit(Vx,Vy)
```

where Vx, Vy are the initial velocities of the cue ball (the magnitude is arbitrary – in fact the code will scale your velocity to have unit magnitude for convenience in doing the animation).

Running the code with potit(-1,-0.81331) sinks the black.