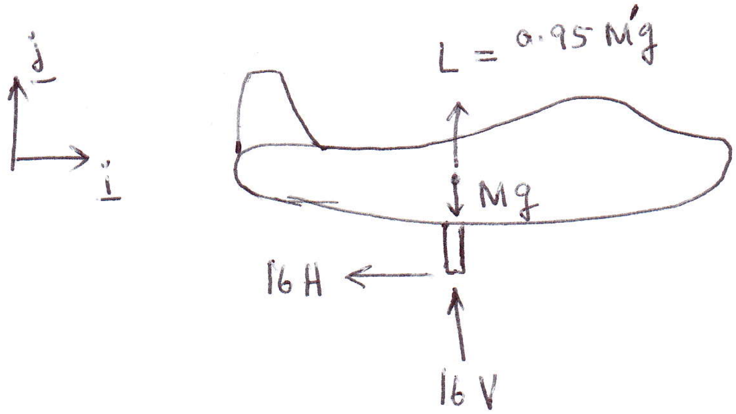
EN40 : Homework #8

① Landing speed $v_0 = 260\ km/hr = 72.2\ m/s$

FBD of aircraft without the wheels



M' : Total mass of aircraft $= 400\times10^3$ kg

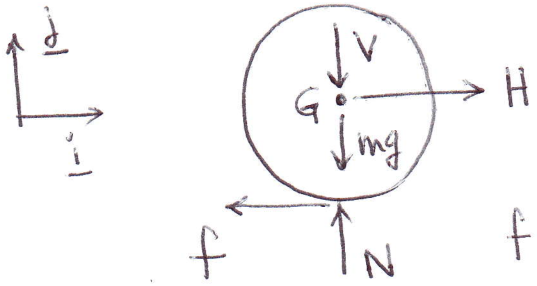
L : lift

M : Mass of aircraft (not including wheels)

H : horizontal reaction force between wheel and axle. There are 16 wheels

V : Vertical reaction force between wheel & axle.

FBD of wheel



m : mass of a single wheel

$f = \mu_k N$ during slip

Aircraft : $\quad 16V + 0.95M'g = Mg \Rightarrow V = (M - 0.95M')\frac{g}{16}$ — (1)

$$-16H = Ma_x \Rightarrow a_x = -\frac{16H}{M} \quad \text{or } H = -\frac{M}{16}a_x \quad \text{— (2)}$$

Wheel : $\quad N - V - mg = 0 \Rightarrow N = V + mg = \left(\frac{M - 0.95M'}{16} + m\right)g$ — (3)

$$H - f = ma_x \Rightarrow -f = \left(\frac{M}{16} + m\right)a_x \quad \text{— (4)}$$

$$-fR = I_G\ddot{\theta} \Rightarrow \ddot{\theta} = -\frac{fR}{I_G} \quad \text{— (5)}$$

Note : we used (1) & (2) to get (3) & (4) .

Let time $t=0$ be the instant of touchdown. The velocity at a subsequent time t is

$$v(t) = v_0 + a_x t \quad \text{——— (6)}$$

Also, from eq.(5), $\dot{\theta}(t) = -\dfrac{fR}{I_G} t$ (note: $\dot{\theta}(0)=0$ The wheel is not rotating initially)

→ (7)

When the aircraft just touches down, there is slip between the wheel and the runway. Slip occurs until $v(t) + R\dot{\theta}(t) > 0$

Rolling without slip commences when $v(t) + R\dot{\theta}(t) = 0$.

Let t^* be the time when this happens.

$$v_0 + a_x t^* - \frac{fR^2}{I_G} t^* = 0 \Rightarrow \boxed{t^* = \frac{v_0}{\dfrac{fR^2}{I_G} - a_x}} \quad \text{——— (8)}$$

Total mass of the aircraft $(M') = M + 16m = 400 \times 10^3$ kg

$$M = 400\times 10^3 - 16\times 125 = 398\,000 \text{ kg}$$

From (3)

$$N = \left(\frac{M - 0.95M'}{16} + m\right) g = 1250g = 12262.5 \text{ N}$$

$$f = \mu_k N = 0.5N = 6131.25 \text{ N}$$

From (4)

$$a_x = -\frac{f}{\left(\dfrac{M}{16} + m\right)} = -\frac{6131.25}{25000} \text{ m/s}^2 = -0.245 \text{ m/s}^2$$

$I_G = m k^2 = 125 \times 0.75^2 \ kg\,m^2 = 70.3 \ kg\,m^2$

$$\frac{fR^2}{I_G} = \frac{6131.25 \times 1^2}{70.3} = 87.2 \ m/s^2$$

$\Rightarrow$ from (8), $\boxed{t^* = \frac{72.2}{87.2 + 0.245} = 0.826\,s}$ skidding time

skidding distance $\boxed{s = v_0 t^* + \frac{1}{2} a_x t^{*2} = 59.6\,m}$

(i) $s = 59.6\ m$

(ii) $t^* = 0.826\ s$

(iii) Deceleration $= -a_x = 0.245\ m/s^2$

(iv) Rate of energy dissipation due to friction $P_f(t) = f\, v_s(t)$

where $v_s(t)$ is the slip speed, $v_s(t) = v(t) + R\dot{\theta}(t)$

$$= v_0 + a_x t - \frac{fR^2}{I_G} t$$

Total energy dissipation due to friction

$$W_f = \int_0^{t^*} P_f(t)\, dt = \int_0^{t^*} f\left[v_0 + a_x t - \frac{fR^2}{I_G} t\right] dt$$

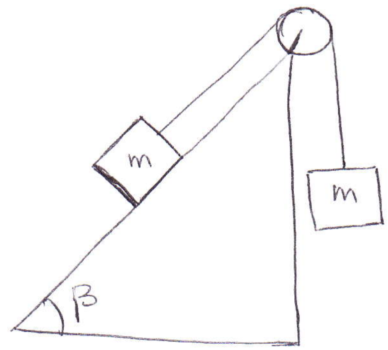
$$= 182\,750\ J$$

This energy is dissipated as heat; it increases the temperature of the tire surface and the rubber residue on the runway, which burn and produce the smoke.

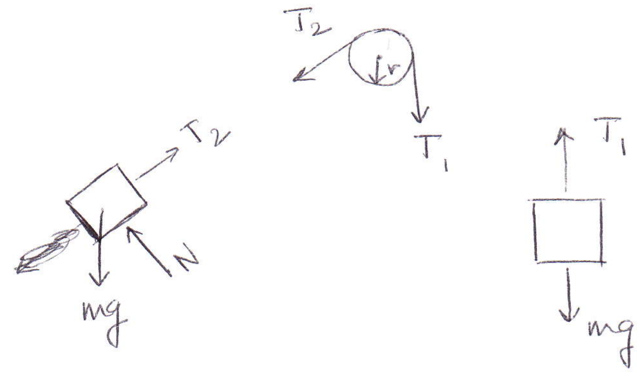
Note that the above is the energy dissipated at ~~the~~ ~~a~~ one wheel only. There are 16 wheels; hence, the total energy dissipated as frictional heat

$$(W_f)_{total} = 16 \times 182750 \text{ J} = 2.924 \times 10^6 \text{ J}$$

(2)



Free body diagrams



String is inextensible ⇒ magnitude of acceleration is the same for both masses. Let the downward acceleration of the right side mass be a.

$$\Rightarrow \quad mg - T_1 = ma$$

$$T_2 - mg\sin\beta = ma$$

$$T_1 r - T_2 r = I_G \alpha$$, where α = angular acceleration of the pulley in the clockwise direction.

String rolls without slipping

$$\Rightarrow \quad a = r\alpha$$

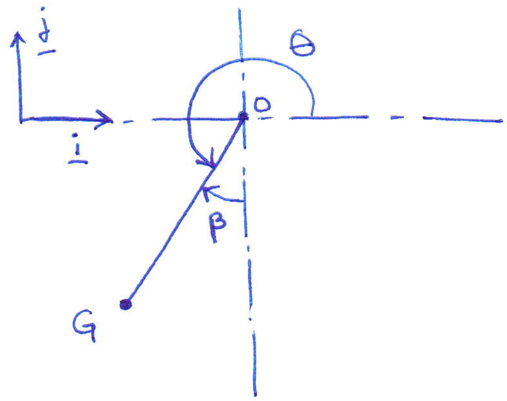
4 equations for 4 unknowns: T_1, T_2, a, α. Solve.

$$\boxed{\alpha = \frac{mgr(1-\sin\beta)}{I_G + 2mr^2}} , \quad \boxed{a = \frac{g(1-\sin\beta)}{\left(\frac{I_G}{mr^2} + 2\right)}}$$

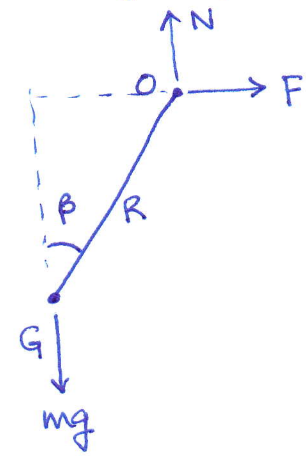
$$\boxed{T_1 = mg\left[\frac{1+\sin\beta + I_G/mr^2}{2 + I_G/mr^2}\right]}$$

$$\boxed{T_2 = mg\left[\sin\beta + \frac{1-\sin\beta}{2+I_G/mr^2}\right]} = mg\left[\frac{1 + (1 + I_G/mr^2)\sin\beta}{2 + I_G/mr^2}\right]$$

(3)



Free body diagram



(i)

F=ma in x-direction

$$F = m\ddot{x}_G \quad \text{--- (1)}$$

F=ma in y-direction

$$N - mg = m\ddot{y}_G \quad \text{--- (2)}$$

Rotational equation of motion about G

$$NR\sin\beta - FR\cos\beta = I_G\ddot{\theta} \quad \text{--- (3)}$$

$$\theta + \beta = \frac{3\pi}{2}$$

$$\Rightarrow \theta = \frac{3\pi}{2} - \beta$$

$$\sin\theta = -\cos\beta$$

$$\cos\theta = -\sin\beta$$

$$\ddot{\theta} = -\ddot{\beta}, \dot{\theta} = -\dot{\beta}$$

In addition to the above 3 equations of motion, we have 2 kinematic equations the relate the acceleration of O to that of G.

$$\ddot{x}_G = \ddot{x}_O - R\ddot{\theta}\sin\theta - R\dot{\theta}^2\cos\theta$$

$$\ddot{y}_G = \ddot{y}_O + R\ddot{\theta}\cos\theta - R\dot{\theta}^2\sin\theta$$

We know that $\ddot{x}_O = a$ and $\ddot{y}_O = 0$

$$\Rightarrow \ddot{x}_G = a + R\ddot{\theta}\cos\beta + R\dot{\theta}^2\sin\beta \quad \text{--- (4)}$$

$$\ddot{y}_G = -R\ddot{\theta}\sin\beta + R\dot{\theta}^2\cos\beta \quad \text{--- (5)}$$

Substitute (1) & (2) in (3)

$$\left(mg + m\ddot{y}_G\right) R \sin\beta \;-\; m\ddot{x}_G\, R\cos\beta \;=\; I_G\,\ddot{\theta}$$

Now, substitute (4) & (5) in the above equation

$$\left(g \cancel{+} - R\ddot{\theta}\sin\beta + R\dot{\theta}^2\cos\beta\right) R\sin\beta - \left(a + R\ddot{\theta}\cos\beta + R\dot{\theta}^2\sin\beta\right) R\cos\beta = \frac{I_G}{m}\ddot{\theta}$$

$$gR\sin\beta - R^2\sin^2\beta\,\ddot{\theta} + \cancel{R^2\cos\beta\sin\beta\,\dot{\theta}^2} - aR\cos\beta - R^2\cos^2\beta\,\ddot{\theta} - \cancel{R^2\sin\beta\cos\beta\,\dot{\theta}^2} = \frac{I_G}{m}\ddot{\theta}$$

$$R\left[g\sin\beta - a\cos\beta\right] - R^2\ddot{\theta} = \frac{I_G}{m}\ddot{\theta}$$

~~or~~ $$\ddot{\theta}\left[I_G + mR^2\right] = mR\left[g\sin\beta - a\cos\beta\right]$$

$$\ddot{\theta} = -\ddot{\beta}$$

$$\Rightarrow \quad \boxed{\ddot{\beta} = \frac{mR}{I_G + mR^2}\left[a\cos\beta - g\sin\beta\right]} \qquad \text{——— (6)}$$

Note that $I_G + mR^2 = I_O$ moment of inertia about point O.

(ii) ~~Multiply both sides of equation (6) with~~ You can proceed in more than one way to get $\dot{\beta}$ from eq. (6).

$$\ddot{\beta} = \frac{d\dot{\beta}}{dt} = \frac{d\dot{\beta}}{d\beta}\cdot\frac{d\beta}{dt} = \frac{d\dot{\beta}}{d\beta}\,\dot{\beta}$$

$$\Rightarrow \quad \ddot{\beta}\,d\beta = \dot{\beta}\,d\dot{\beta}$$

Let's integrate this equation from the initial state $(t=0)$ where $\beta = 0$ and $\dot{\beta} = 0$ (starting from rest).

$$\int_0^{\beta} \ddot{\beta}\, d\beta = \underbrace{\int_0^{\dot{\beta}} \dot{\beta}\, d\dot{\beta}}_{= \frac{1}{2}\dot{\beta}^2}$$

$$\Rightarrow \frac{1}{2}\dot{\beta}^2 = \int_0^{\beta} \ddot{\beta}\, d\beta = \frac{mR}{I_o} \int_0^{\beta} (a \cos\beta - g \sin\beta)\, d\beta$$

$$= \frac{mR}{I_o} \Big[a \sin\beta + g \cos\beta \Big]_0^{\beta}$$

$$= \frac{mR}{I_o} \Big[a \sin\beta + g \cos\beta - g \Big]$$

$$= \frac{mR}{I_o} \Big[a \sin\beta - g (1 - \cos\beta) \Big]$$

$$\Rightarrow \boxed{\dot{\beta}^2 = \frac{2mR}{I_o} \Big[a \sin\beta - g (1 - \cos\beta) \Big]} \quad \text{———} \; (7)$$

When β reaches the maximum value, $\frac{d\beta}{dt} = \dot{\beta} = 0$.

$$\Rightarrow \boxed{a \sin\beta + g \cos\beta - g = 0} \quad \text{———} \; (8)$$

100 km/hr = 27.8 m/s

0 – 100 km/hr in 10 s $\Rightarrow a = 2.78 \text{ m/s}^2$.

Solve equation (8) for β, with $a = 2.78 \text{ m/s}^2$ and $g = 9.81 \text{ m/s}^2$

[Use MAPLE or MATHEMATICA or MATLAB ("fsolve") or Calculator, ...]

$$\beta_{max} = 0.552 \text{ rad} = \boxed{31.64^\circ}$$

(iii) $m = 1\ kg$, $I_G = 0.01\ kg\,m^2$, $R = 0.2\,m$.

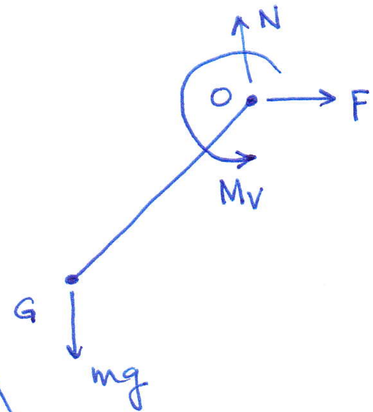
At $t = 0$, $\beta = 0$, $\dot{\beta} = 0$.

We need to plot $\beta(t)$, $F(t)$, $N(t)$.

Here is our strategy. Let's first integrate eq(6) and get $\beta(t)$ and $\dot{\beta}(t)$. Then we will use the solution in eqs.(1)-(5) to get $F(t)$ and $N(t)$.

But first, we need to account for the viscous moment M_v.

With the viscous moment, eq.(3) can be written as



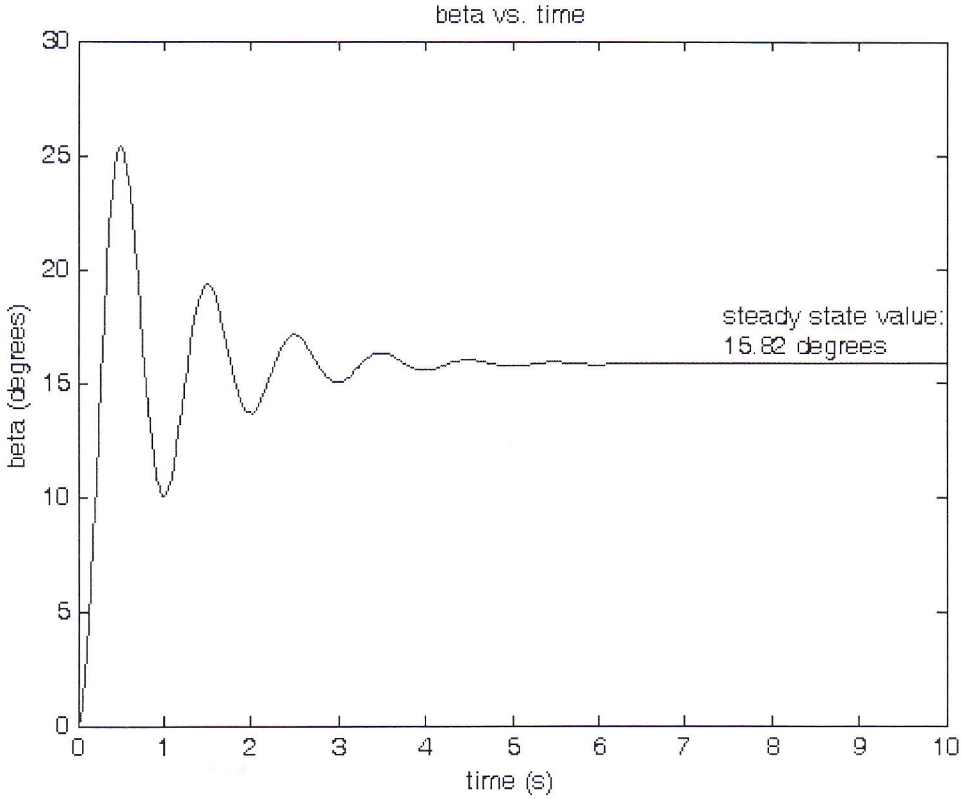
$$N R \sin\beta - F R \cos\beta + M_v = I_G \ddot{\Theta}$$

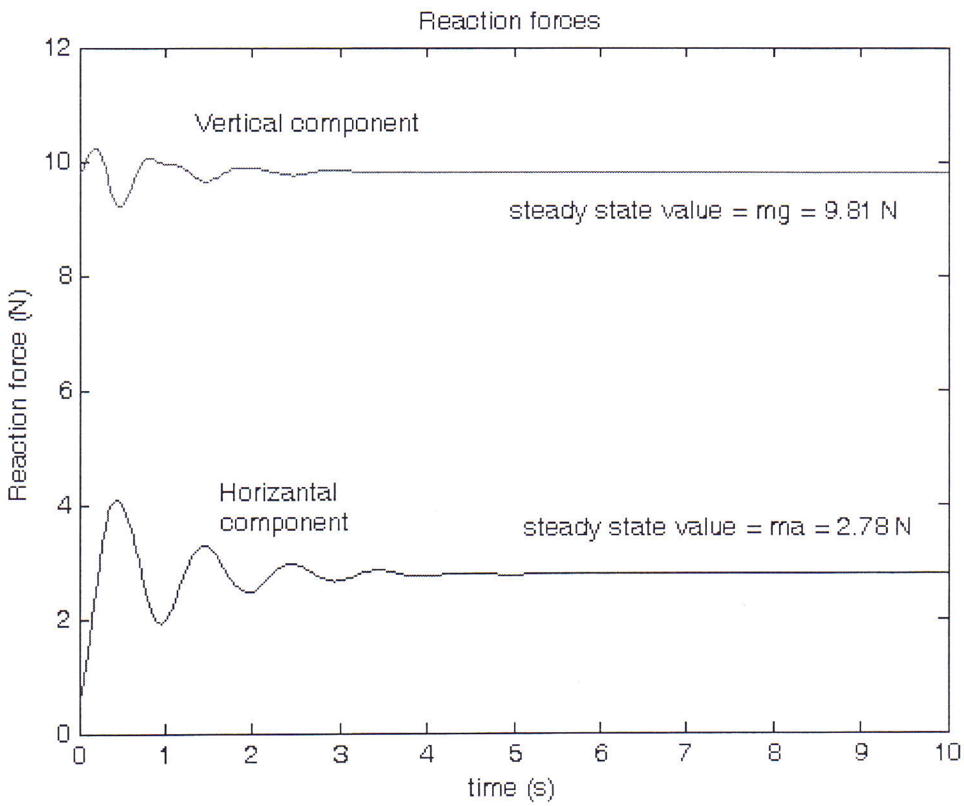
Now, eq.(6) becomes

$$\boxed{\begin{aligned} \ddot{\beta} &= \frac{mR}{I_G + mR^2}\left[a\cos\beta - g\sin\beta\right] - \frac{M_v}{(I_G + mR^2)} \\ &\text{where } M_v = c\,\dot{\beta} \end{aligned}} \quad \text{—— (9)}$$

See the MATLAB code and plots.

```
function rigid_pendulum

    close all
    m = 1.0; % mass of the pendulum
    IG = 0.01; % moment of inertia of the pendulum about the center of mass
    R = 0.2; % length of the pendulum
    g=9.81; % acceleration due to gravity
    c=0.1;
    a = 2.78; % acceleration of the bus

    w0=[0,0]; % initial values of beta and beta-dot
    time=10; % integration time
    [tv,wv]=ode45(@eom,[0,time],w0);

    plot(tv,wv(:,1)*180/pi); % plot the value of beta with time
    wv(length(tv),1)*180/pi % print the final value of bea to the screen

    % this for loop is to calculate the horizantal and vertical reaction
    % forces at the pin joint, F and N respectively. First, use eq. 9 in
    % eqs. 4&5 to get the x and y accelerations of G. Then use Eqs. 4 and 5
    % in Eqs. 1 and 2 to get the reaction forces.
    for i=1:length(tv)
       beta=wv(i,1); betadot=wv(i,2);
       b2dot=m*R/(IG+m*R^2)*(a*cos(beta)-g*sin(beta))-c/(IG+m*R^2)*betadot;
       xG2dot=a-R*b2dot*cos(beta)+R*betadot^2*sin(beta);
       yG2dot=R*b2dot*sin(beta)+R*betadot^2*cos(beta);
       F(i)=m*xG2dot;
       N(i)=m*g+m*yG2dot;
    end

    figure
    plot(tv,F) % plot the horizantal force
    hold on
    plot(tv,N,'r') % plot the vertical force
    hold off

    function dwdt=eom(t,w)
        beta=w(1); betadot=w(2);
        beta2dot=m*R/(IG+m*R^2)*(a*cos(beta)-g*sin(beta))-c/(IG+m*R^2)*betadot;
        dwdt=[betadot;beta2dot];
    end
end
```


beta vs. time
beta (degrees)
time (s)
steady state value:
15.82 degrees
0
5
10
15
20
25
30
0
1
2
3
4
5
6
7
8
9
10

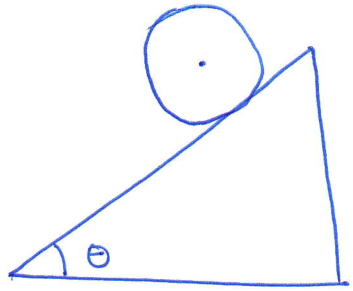

Reaction forces
Reaction force (N)
time (s)
Vertical component
steady state value = mg = 9.81 N
Horizantal
component
steady state value = ma = 2.78 N
0
2
4
6
8
10
12
0
1
2
3
4
5
6
7
8
9
10

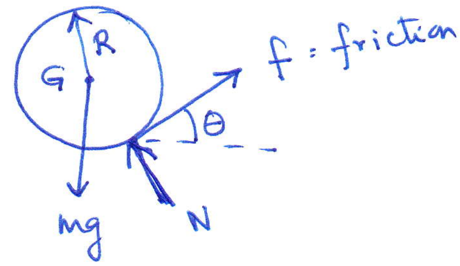
④ Rolling disk

Free body diagram

$\mu_s = 0.15$

$\mu_k = 0.1$





$$\left.\begin{aligned} &\text{Normal to the inclined plane:} && N - mg\cos\theta = 0 \\ &\text{Parallel to the inclined plane:} && mg\sin\theta - f = ma \\ &\text{Rotation:} && fR = I_G\,\alpha \end{aligned}\right]$$

\# of unknowns N, f, a, α. Need one more equation

Two possibilities at contact point

(a) Rolling without slipping : $a = R\alpha$ [check if $f < \mu_s N$]

(b) Sliding : $f = \mu_k N$

Assume rolling without slipping. $m = 5\text{ kg}$, $R = 0.15\text{ m}$, $\theta = 30^\circ$.

Solve 4 equations for ~~for~~ 4 unknowns

$$\left.\begin{aligned} N &= mg\cos\theta \\ f &= mg\left[\frac{I_G\sin\theta}{I_G + mR^2}\right] \end{aligned}\right\} \quad \frac{f}{N} = \frac{I_G\tan\theta}{I_G + mR^2}$$

For the disk, $I_G = \dfrac{mR^2}{2}$

$$a = g\left[\frac{\sin\theta}{1 + I_G/mR^2}\right] \qquad = \frac{\frac{1}{2}\tan\theta}{\frac{3}{2}} = \frac{\tan\theta}{3}$$

$$\alpha = \frac{g}{R}\left[\frac{\sin\theta}{1 + I_G/mR^2}\right] \qquad = \frac{\tan 30^\circ}{3} = 0.192 > \mu_s$$

$\Rightarrow$ This solution is not valid.

So, adopt the other possibility $f = \mu_k N$. Solve 4 equations for 4 unknowns.

$$a = g(\sin\theta - \mu_k \cos\theta) = 4.06\ m/s^2 \longrightarrow \text{(i)}$$

$$\alpha = \frac{mgR}{I_G} \mu \cos\theta = 11.33\ rad/s^2$$

$$\cancel{f} \quad N = mg \cos\theta = \cancel{4.248}\ 42.48\,N$$

$$f = \mu_k mg \cos\theta = \cancel{42.48\,N}\ 4.248\,N \longrightarrow \text{(ii)}$$

(iii) For $\theta = 10^\circ$, $\frac{\tan\theta}{3} = 0.059 < \mu_s$.

$\Rightarrow$ The assumption of rolling without slip is valid. So, we can use the solution from the previous page.

$$N = mg \cos\theta = \cancel{42.48\,N}\ 48.3\,N$$

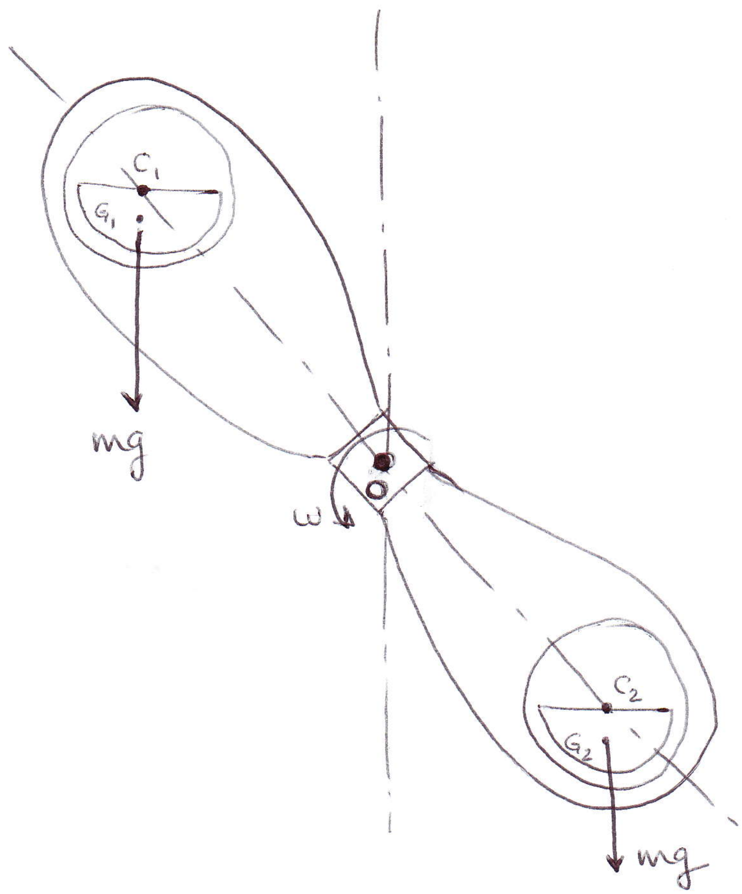
$$f = mg\left[\frac{I_G \sin\theta}{I_G + mR^2}\right] = \cancel{8.18\,N}\ 2.84\,N$$

$$a = g\left[\frac{\sin\theta}{1 + I_G/mR^2}\right] = 1.14\ m/s^2$$

$$\alpha = \frac{g}{R}\left[\frac{\sin\theta}{1 + I_G/mR^2}\right] = 7.57\ rad/s^2$$

⑤ (i) $\omega = \dfrac{\pi}{5.5 \times 60} \dfrac{rad}{s} = 0.00952 \ rad/s$

(ii)



Consider the wheel at some point during its operation as shown above. The center of mass of each load is directly below its center of rotation, i.e. G_1 is always 1m below C_1 and G_2 is always 1m below C_2. So the velocity of G_1 is the same as that of C_1; the velocity of G_2 is the same as that of C_2.

The moment due to the weights of ~~bo~~ the two loads cancel each other. So during the constant ω phase, the only work done by the motor is that dissipated in bearings, friction between gear teeth, etc. Since we are ignoring these losses, the only other work done by the motor is to accelerate the wheel from $\omega = 0$ to $\omega = 0.00952$ rad/s.

The kinetic energy of the rotating wheel = sum of the individual kinetic energies of the two loads on either side. Treating each load as a point mass at its center of mass.

$$KE = \frac{1}{2} m v_{G1}^2 + \frac{1}{2} m v_{G2}^2$$

$$= \frac{1}{2} m v_{C1}^2 + \frac{1}{2} m v_{C1}^2$$

$$= \frac{1}{2} m R^2\omega^2 + \frac{1}{2} m R^2\omega^2 = m R^2 \omega^2$$

$$= 300 \times 10^3 \times 5.5^2 \times 0.00952^2 \text{ J}$$

$$= 822.47 \text{ J}$$

(iii) 1 Unit of electricity = 1 kWhr = $10^3 \times 3600 = 3.6 \times 10^6$ J.

So, 822.47 J = 2.285×10^{-4} kWhr → negligible.

Since there are two counter-balancing loads, one on each side, it takes very little energy to operate the Falkirk wheel. The wheel is operated by just a 30hp motor, which consumes about 2 kwhr (mostly bearing friction etc.) per each half cycle!