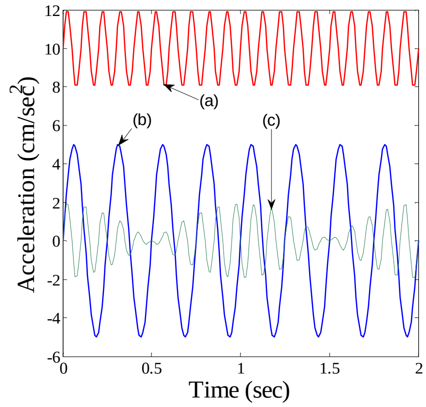
School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 2: Dynamics of Particles
Due Friday Feb 11, 2011
MAX SCORE 65 POINTS

1. The figure shows three measurements from an accelerometer.



1.1 Which signal can be approximated as simple harmonic motion, and why?

Signal (b) is SHM. (a) has a nonzero mean and (c) has a variable amplitude. **[1 POINT]**

1.2 Determine the amplitude of the velocity and displacement for the signal which can be approximated as simple harmonic motion.

For signal (b) the acceleration has amplitude 5 cm/s^2. There are 8 cycles in 2 sec, so the period is 0.25s. The (angular) frequency follows as $\omega = 2\pi / T = 8\pi \;\; \text{rad}\,\text{s}^{-1}$. For simple harmonic motion, the displacement, velocity and acceleration have the form

$$x(t) = X_0 \sin\omega t \qquad v(t) = V_0 \cos\omega t \qquad a(t) = -A_0 \sin\omega t$$

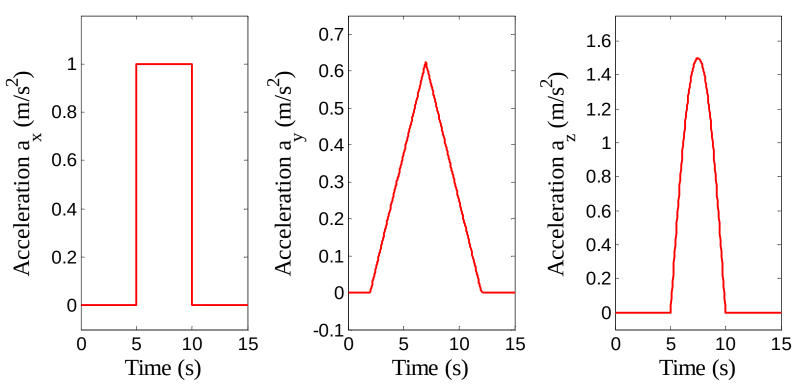
$$V_0 = \omega X_0 \qquad A_0 = \omega V_0$$

where X_0, V_0, A_0 are the amplitudes of the displacement, velocity and acceleration. The velocity and displacement amplitudes follow as

$$V_0 = A_0 / \omega = 5/(8\pi) = 0.199\text{cm s}^{-1} \quad X_0 = A_0/\omega^2 = 5/(8\pi)^2 = 0.0079\text{cm}$$

[2 POINTS]

2. A three-axis accelerometer mounted on an inertial platform with fixed orientation measures three components of acceleration shown in the figure. The vertical acceleration component a_z can be approximated as $a_z = A\sin(2\pi(t-t_0)/T)$ while it is nonzero, with an appropriate choice of A, t_0 and T. If the platform is initially at rest, what is its velocity and speed after 15 sec?



The velocity components can be determined by integrating the accelerations. For the x and y components, we can simply find the areas under the curves, so that

$v_x = 5m/s \qquad v_y = 0.625 \times 10/2 = 3.125 m/s$ **[2 POINTS]**

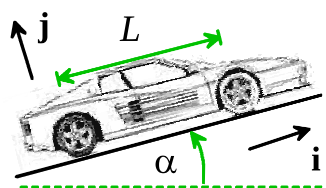
The z component must be found by integration. We must first identify the values of A, t_0 and T. From the graph $A = 1.5ms^{-2}$ $T = 10\,\text{sec}$ $t_0 = 5\,\text{sec}$. Then

$$v_z = \int_5^{10} 1.5\sin(2\pi(t-5)/10)dt = 4.775 m/s \quad \textbf{[2 POINTS]}$$

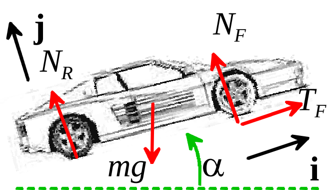
The velocity is therefore $\mathbf{v} = (5\mathbf{i} + 3.125\mathbf{j} + 4.775\mathbf{k})m/s$ and the speed is the magnitude of this vector, i.e. $|\mathbf{v}| = \sqrt{5^2 + 3.125^2 + 4.775^2} = 7.587 m/s$.

[1 POINT]

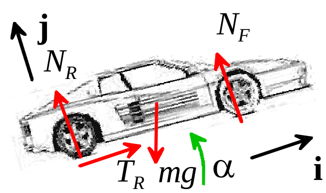
3. The figure shows a vehicle climbing a slope of angle α. The interface between tyres and road has friction coefficient μ. The center of mass of the vehicle is midway between the two wheels, and a distance h above the ground.



3.1 Draw a free body diagram showing the forces acting on the car. Assume that the vehicle has front wheel drive, and that the rear wheel rolls freely.



Front wheel drive



Rear wheel drive

[3 points – front wheel drive only for this part]

3.2 Suppose that the driver applies the maximum possible power so that the driven wheels slip on the road surface. Calculate the acceleration of the vehicle.

Newton's law for the front-wheel drive vehicle $(T_F - mg\sin\alpha)\mathbf{i} + (N_R + N_F - mg\cos\alpha)\mathbf{j} = a_x\mathbf{i}$

[1 POINT]

Moments about the COM must vanish. $(N_F - N_R)L/2 + T_F h = 0$

[1 POINT]

If the wheel spins, then $T_F = \mu N_F$

[1 POINT]

Solve four equations for four unknowns

```
eq1 := Tf-m*g*sin(alpha)=m*ax:
eq2 := Nr+Nf-m*g*cos(alpha)=0:
eq3 := (Nf-Nr)*L/2+Tf*h=0:
eq4 := Tf=mu*Nf:
solve({eq1,eq2,eq3,eq4},{Nr,Nf,Tf,ax});
```

$$\left\{Nf = \frac{1}{2}\frac{L\,m\,g\cos(\alpha)}{L+\mu h},\ Nr = \frac{1}{2}\frac{m\,g\cos(\alpha)(L+2\mu h)}{L+\mu h},\ ax = \frac{1}{2}\frac{g(\mu\cos(\alpha)L - 2\sin(\alpha)L - 2\sin(\alpha)\mu h)}{L+\mu h},\ Tf = \frac{1}{2}\frac{\mu L\,m\,g\cos(\alpha)}{L+\mu h}\right\}$$

$$a_x = g\,\frac{\mu(L\cos\alpha - 2h\sin\alpha) - 2L\sin\alpha}{2(L+\mu h)}$$

[2 POINTS]

3.3 Repeat 3.1 and 3.2 for a rear-wheel drive vehicle. Which gives a greater acceleration (front or rear wheel drive)? **[3 POINTS FOR FBD]**

Newton's law for the front-wheel drive vehicle $(T_R - mg\sin\alpha)\mathbf{i} + (N_R + N_F - mg\cos\alpha)\mathbf{j} = a_x\mathbf{i}$

[1 POINT]

Moments about the COM must vanish. $(N_F - N_R)L/2 + T_R h = 0$

[1 POINT]

If the wheel spins, then $T_R = \mu N_R$

[1 POINT]

```
eq1 := Tr-m*g*sin(alpha)=m*ax:
eq2 := Nr+Nf-m*g*cos(alpha)=0:
eq3 := (Nf-Nr)*L/2+Tr*h=0:
eq4 := Tr=mu*Nr:
solve({eq1,eq2,eq3,eq4},{Nr,Nf,Tr,ax});
```

$$\left\{Nf = \frac{1}{2}\,\frac{m\,g\cos(\alpha)\,(L-2\,\mu\,h)}{L-\mu\,h},\ Nr = \frac{1}{2}\,\frac{L\,m\,g\cos(\alpha)}{L-\mu\,h},\ Tr = \frac{1}{2}\,\frac{\mu\,L\,m\,g\cos(\alpha)}{L-\mu\,h},\ ax = \frac{1}{2}\,\frac{g\,(\mu\cos(\alpha)\,L - 2\sin(\alpha)\,L + 2\sin(\alpha)\,\mu\,h)}{L-\mu\,h}\right\}$$

$$a_x = g\,\frac{\mu(L\cos\alpha + 2h\sin\alpha) + 2L\sin\alpha}{2(L-\mu h)}$$

[2 POINTS]

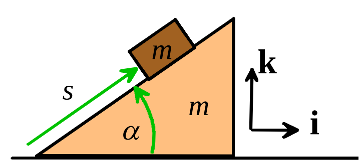
Subtracting the first solution from the second gives

$$a_{xR} - a_{xF} = \frac{g\mu^2 Lh\cos\alpha}{L^2 - \mu^2 h^2}$$

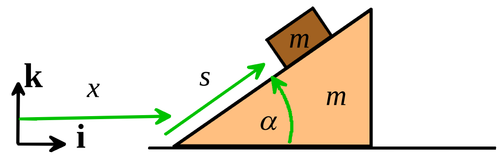
Note that for $N_f > 0$ (i.e. the front wheel does not leave the ground) $L > 2\mu h$, so $L^2 - \mu^2 h^2 > 0$. Rear wheel drive therefore has the greater acceleration – the difference gets smaller as the slope increases.

[1 POINT]

4. The figure shows a block of mass *m* on the surface of a wedge with identical mass. The interface between wedge and block has friction coefficient $\mu = \beta\tan\alpha$, where $0 < \beta < 1$. The wedge is supported by a frictionless horizontal surface. At time *t=0* the system is at rest, with the block a distance *s=d* from the base of the wedge. Find an expression for the time required for the block to reach the base of the wedge. Your solution should include the following steps



4.1 Let a_x denote the horizontal acceleration of the wedge, and let *s* denote the distance of the block from the base at some arbitrary time *t*. . Write down the acceleration vectors for the wedge and block in terms of these variables (and their time derivatives)



The picture shows the situation after the wedge has slid by some distance x, and the block is a distance s from the base. The acceleration vector for the wedge is $\mathbf{a} = a_x \mathbf{i}$. For the block, we have

$$\mathbf{r} = x\mathbf{i} + s\cos\alpha\,\mathbf{i} + s\sin\alpha\,\mathbf{k}$$

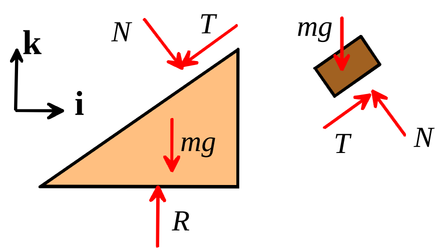
$$\mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{ds}{dt}(\cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{k})$$

$$\mathbf{a} = a_x\mathbf{i} + \frac{d^2 s}{dt^2}(\cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{k})$$

[2 POINTS]

4.2 Draw free body diagrams for both the block and the wedge

Free body diagrams are shown



It is essential to draw the friction force in the correct direction, since the block is slipping on the surface of the wedge.

[3 POINTS]

4.3 Write down Newton's laws of motion for the block and the wedge

For the block, we have

$$(T\cos\alpha - N\sin\alpha)\mathbf{i} + (N\cos\alpha + T\sin\alpha - mg)\mathbf{k} = m\left[a_x\mathbf{i} + \frac{d^2 s}{dt^2}(\cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{k})\right]$$

For the wedge

$$(-T\cos\alpha + N\sin\alpha)\mathbf{i} + (R - mg - N\cos\alpha - T\sin\alpha)\mathbf{k} = ma_x\mathbf{i}$$

[3 POINTS]

4.4 Hence, calculate a formula for $d^2 s/dt^2$, and hence find a formula for s as a function of time. The result will give the time.

The friction law relates T and N: $T = \mu N = \beta\tan\alpha N$

[1 POINT]

This gives us five equations in five unknowns – solving for $d^2 s/dt^2$

```
restart:
eq1 := T*cos(alpha)-N*sin(alpha)=m*(ax+d2sdt2*cos(alpha)):
eq2 := N*cos(alpha)+T*sin(alpha)-m*g=m*d2sdt2*sin(alpha):
eq3 := -T*cos(alpha)+N*sin(alpha)=m*ax:
eq4 := R-m*g-N*cos(alpha)-T*sin(alpha)=0:
eq5 := T=beta*tan(alpha)*N:
d2sdt2 := subs(solve({eq1,eq2,eq3,eq4,eq5},{T,N,ax,d2sdt2,R}),d2sdt2);
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$$d2sdt2 := \frac{2\,g\,\sin(\alpha)\,(\beta-1)}{-\cos(\alpha)^2-\beta+2+\cos(\alpha)^2\,\beta}$$

So

$$\frac{d^2s}{dt^2}=-\frac{2g\sin\alpha(1-\beta)}{2-\beta+(\beta-1)\cos^2\alpha}=-\frac{2g\sin\alpha(1-\beta)}{1+(1-\beta)\sin^2\alpha}$$

[2 POINTS]

The acceleration is constant, so we can calculate the time using the constant acceleration straight-line motion formulas

$$s=d-\frac{g\sin\alpha(1-\beta)}{2-\beta+(\beta-1)\cos^2\alpha}t^2$$

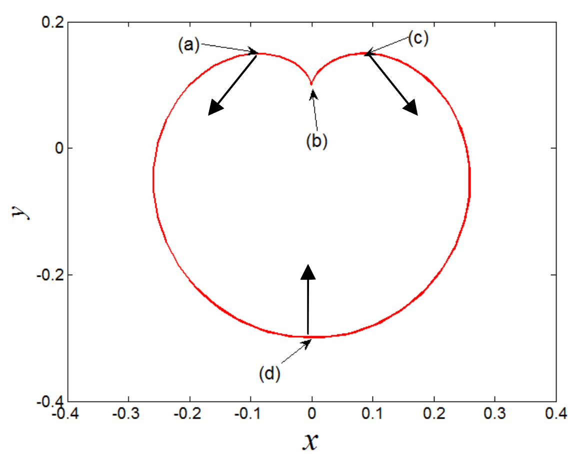
Therefore, setting $s=0$, we get

$$t=\sqrt{\frac{gd\sin\alpha(1-\beta)}{2-\beta+(\beta-1)\cos^2\alpha}}$$ (note the other possible formula with sin also…)

[2 POINTS]

5. The figure shows the trajectory of a charged particle in a ‘Penning Trap’, for a particular choice of the electric and magnetic fields that trap the particle. The particle remains in the (x,y) plane at all times, and move from a to b to c to d…

- At point (a), the particle’s speed is decreasing
- At point (c), the particle’s speed is increasing
- At point (d), the particle’s speed is a maximum



5.1 Draw arrows on a copy of the figure at points (a), (c), and (d) to show the approximate direction of the particle’s acceleration vector.

[3 POINTS – 1 each]

5.2 What is the particle’s speed at point (b)?

The horizontal component of velocity is zero, because the trajectory is vertical. In addition, the direction of motion reverses, so the particle must come to rest at *b*. Its speed is zero.

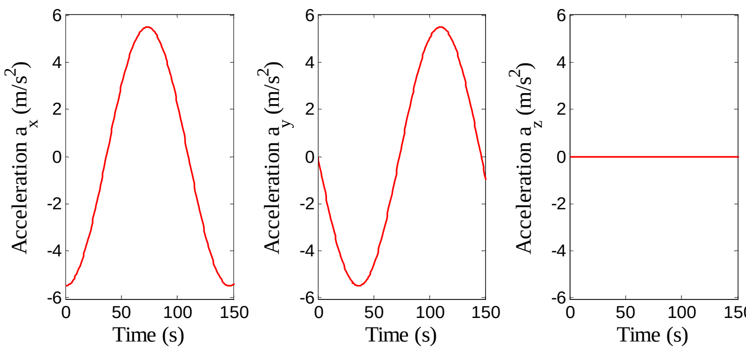
[1 POINT]

5.3 How would the acceleration change if the direction of motion of the particle was reversed?

It would not change at all. (**Be careful when grading this problem – the question is not completely clear. The problem meant that the entire history of motion is reversed, i.e. the particle travels in the opposite direction, and slows at c, and speeds up at (a). Some people may interpret only the direction to be reversed, in which case the accelerations at (a) and (c) switch over. If this is explained it should receive full credit.**

[1 POINT]

6. An aircraft flies at constant altitude around a circular path. Its navigation system includes an inertial platform (a device which maintains a fixed orientation with respect to space, regardless of the motion of the aircraft). The accelerations measured by three mutually perpendicular accelerometers mounted on this platform are shown in the figure.



6.1 Let V denote the aircraft's speed, and let **r**=R**i** denote the position vector of the aircraft at time $t=0$. Find formulas for the position, velocity and acceleration vectors of the aircraft in terms of R, V and t.

Recall that the circular motion formulas are

$$\mathbf{r} = R\cos(Vt/R)\mathbf{i} + R\sin(Vt/R)\mathbf{j}$$

$$\mathbf{v} = -V\sin(Vt/R)\mathbf{i} + V\cos(Vt/R)\mathbf{j}$$

$$\mathbf{a} = -\frac{V^2}{R}\left(\cos(Vt/R)\mathbf{i} + \sin(Vt/R)\mathbf{j}\right)$$

Where we have used the formula $V = R\omega$ and have taken V to be constant so that $\theta = \omega t$ (the accelerations would not look like sin and cos curves if V is changing)

[3 POINTS]

6.2 Hence, use the measurements to determine the radius of the aircraft's path, and the aircraft's speed.

The formulas show that acceleration components are harmonic, with amplitude $\frac{V^2}{R}$ and period $T = 2\pi R./V$. We can read the amplitude and period off the graphs = T=150 sec and A=5.5 m/s^2. We can solve for the unknown speed and radius, with the result $V = 131 m/s \quad R = 3134.6m$ (about 250 knots and 2 miles)

This problem can also be solved in a rather more complicated way by writing down the accelerations and integrating them – for example

$$\mathbf{a} = -5.5\cos(\pi t/75)\mathbf{i} - 5.5\sin(\pi t/75)\mathbf{j}$$

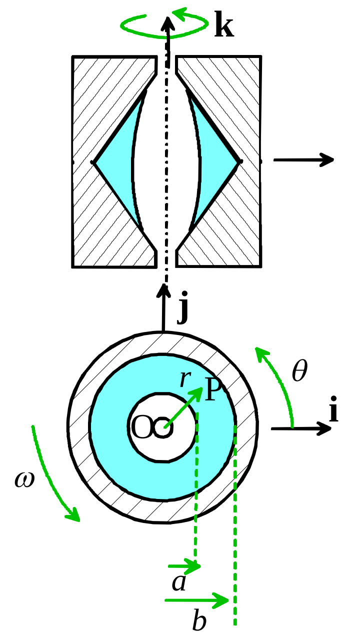
$$\Rightarrow \mathbf{v} = \left(-5.5\sin(\pi t/75)\mathbf{i} + 5.5\cos(\pi t/75)\mathbf{j}\right)75/\pi + \mathbf{v}_0$$

$$\mathbf{r} = (5.5\cos(\pi t/75)\mathbf{i} + 5.5\sin(\pi t/75)\mathbf{j})75^2/\pi^2 + \mathbf{v}_0 t + \mathbf{r}_0$$

where $\mathbf{v}_0, \mathbf{r}_0$ are two constant vectors (arbitrary constants of integration). The path is circular if and only if $\mathbf{v}_0 = 0$, in which case the position vector is a parametric equation for a circle centered at $\mathbf{r}_0$. The radius of the circle is $5.5 \times 75^2/\pi^2$ (same ans as before) and the speed is just the magnitude of **v**, i.e. $5.5(75/\pi)$. Either approach should receive full credit.

[2 POINTS]

7. The figure shows a micro-centrifuge (see Marziali *et al* PNAS, 1999 for a detailed description). It is used to extract particles from a fluid suspension.



7.1 Consider a point in the fluid that is a (constant) distance r from the axis of the centrifuge, and subtends an angle $\theta(t)$ to the fixed $\mathbf{i}$ direction as shown in the figure. The point spins with the fluid at constant angular speed $d\theta / dt = \omega$. Write down the position vector of the point, and hence determine its velocity and acceleration vector, expressing your answer as components in the fixed $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ basis.

$$\mathbf{r} = r\cos(\omega t)\mathbf{i} + r\sin(\omega t)\mathbf{j}$$
$$\mathbf{v} = -\omega r\sin(\omega t)\mathbf{i} + r\omega\cos(\omega t)\mathbf{j} \quad \textbf{[2 POINTS]}$$
$$\mathbf{a} = -r\omega^2(\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j})$$

7.2 Write down the components of a unit vector parallel to the (rotating) line OP. Hence, find an expression for the component of acceleration parallel to OP (the radial acceleration).

$$\mathbf{n} = \cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}$$
$$a_r = \mathbf{a}\cdot\mathbf{n} = -r\omega^2$$

[2 POINTS]

7.3 The radial velocity of a fluid particle suspended in the fluid is usually approximated as

$$v_r = -Sa_r$$

where a_r is the radial acceleration of the fluid, and S is a constant (with units of time) called the 'Sedimentation coefficient.' Find an expression for the time for a fluid particle to sediment from position $r=a$ at the surface of the fluid to $r=b$ at the circumference of the centrifuge.

$$v_r = -Sa_r \Rightarrow \frac{dr}{dt} = Sr\omega^2 \Rightarrow \int_a^b \frac{dr}{r} = S\omega^2\int_0^t dt \Rightarrow t = \frac{1}{S\omega^2}\log(b/a)$$

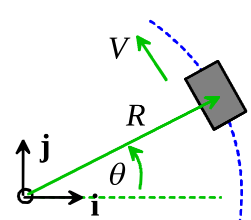
[2 POINTS]

7.4 E Coli has a sedimentation coefficient of 70×10^{-13}s, the centrifuge spins at 100000rpm, and the fluid radii are $a = 0.1$ inches and $b = 0.29$ inches. Determine the time required to sediment E coli from suspension.

$$t = \frac{1}{70\times10^{-13}(2\pi\times10^5/60)^2}\log(0.29/0.1) = 1387s = 23\min$$

[1 POINT]

8. An airport 'people mover' travels at constant speed V around a circular path with radius R. The figure shows a plan view of the path.



8.1 Write down the position vector of the vehicle in terms of R and the angle θ shown in the figure.

$$\mathbf{r} = R\cos\theta\mathbf{i} + R\sin\theta\mathbf{j}$$

[1 POINT]

8.2 Hence, calculate formulae for the velocity and acceleration vectors for the vehicle, in terms of R, V, and θ, expressing your answer as components in the basis shown.

Note that θ varies with time. Differentiate the position vector, using the Chain rule

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -R\sin\theta\mathbf{i}\frac{d\theta}{dt} + R\cos\theta\frac{d\theta}{dt}\mathbf{j} = R\frac{d\theta}{dt}\left(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}\right).$$

Note that $\left(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}\right)$ is a unit vector, so $Rd\theta/dt$ is the magnitude of the velocity (i.e. the speed V) and therefore $R\frac{d\theta}{dt} = V$. Thus

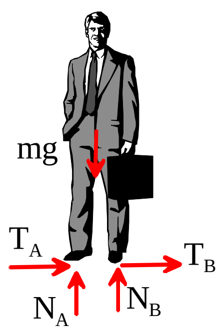
$$\mathbf{v} = V\left(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}\right)$$

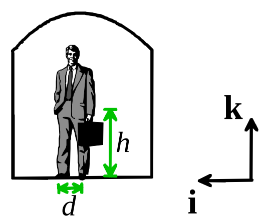
Differentiate again to find the acceleration

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = V\left(-\cos\theta\mathbf{i} - \sin\theta\mathbf{j}\right)\frac{d\theta}{dt} = \frac{V^2}{R}\left(-\cos\theta\mathbf{i} - \sin\theta\mathbf{j}\right)$$

[2 POINTS]

8.3 The figure shows a passenger inside the car, at the instant when $\theta = 0$. His center of mass is a height h above the floor, and he stands with feet a distance d apart, facing in the direction of motion of the vehicle. There is sufficient friction between the floor and his feet to prevent slip. Draw a free body diagram showing the forces acting on the passenger.





[2 POINTS]

8.4 By considering the motion of the passenger at the instant when $\theta = 0$, determine formulae for the reaction forces exerted on the passenger by the floor of the vehicle, in terms of m, g, V, R, d and h. Not all the forces can be determined uniquely.

Substituting $\theta = 0$ into the acceleration formula from 8.2, and writing down Newton's law gives

$$\mathbf{F} = -(T_A + T_B)\mathbf{i} + (N_A + N_B - mg)\mathbf{k} = m\mathbf{a} = -m\frac{V^2}{R}\mathbf{i}$$

We can get another equation of motion by idealizing the passenger as a massless frame, in which case moments about the COM must vanish, i.e.

$$\left(T_A + T_B\right)h\mathbf{j} + (N_B - N_A)\frac{d}{2}\mathbf{j} = \mathbf{0}$$

The $\mathbf{i}$ component of the first equation gives.

$$\left(T_A + T_B\right) = \frac{mV^2}{R}$$

Using the **k** component of the first equation, and substituting for $\left(T_A + T_B\right)$ in the second gives

$$(N_A + N_B - mg) = 0$$

$$\frac{mV^2}{R}h + (N_B - N_A)\frac{d}{2} = 0$$

These can be easily solved to give

$$N_A = \frac{mg}{2} + \frac{mV^2}{R}\frac{h}{d} \qquad N_B = \frac{mg}{2} - \frac{mV^2}{R}\frac{h}{d}$$

[3 POINTS]

8.5 Finally, calculate an expression for the minimum allowable radius of the path for the passenger to remain standing, in terms of *V*, *g*, *h* and *d*.

The passenger tips over if his feet lose contact with the ground. Contact is lost if the reaction force is zero or negative. The preceding part of the problem shows that N_A is always positive, but N_B will be zero if the radius is too small. So

$$N_B = \frac{mg}{2} - \frac{mV^2}{R}\frac{h}{d} > 0 \Rightarrow R > \frac{2V^2}{g}\frac{h}{d}$$

[2 POINTS]