



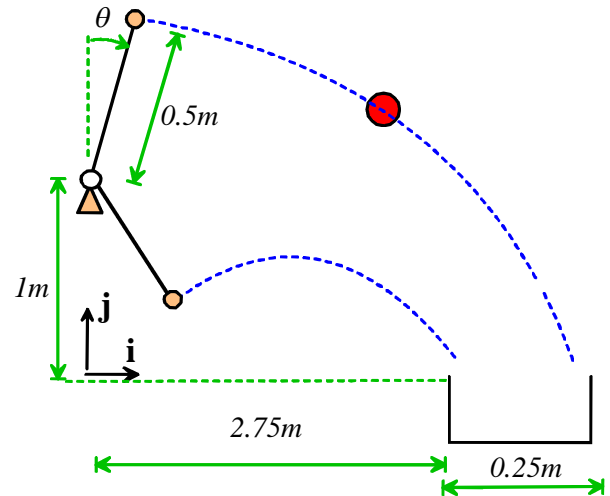
School of Engineering  
Brown University

## EN40: Dynamics and Vibrations

### Homework 3: Solving equations of motion for particles Due Friday Feb 18 2011

1. Suppose that you wish to throw a piece of garbage into a garbage can. Is it better to use an underarm or overarm throw? A [detailed analysis of this important question](#) has been published by Professor Mahadevan and his group at Harvard, (who is evidently putting his MacArthur fellowship to good use). To show that first-year students at Brown are as smart as faculty at Harvard, you will work through a simplified version of their calculations. To keep things simple, we will make the following assumptions:

- You throw by rotating your arm at the elbow, in a seated position. Your elbow has the same height for both under- and over-arm throws
- You throw the projectile with a speed of 5 m/s, for both over- and under-arm throws.
- Relevant dimensions are shown in the figure.



Your garbage will go into the basket only if you release the projectile at the right point. Specifically, there will be some range of release angles  $\theta_1 < \theta < \theta_2$  for which the garbage will fall within the can. If the range is large, the shot is easy; if the range is small, the target is hard to hit.

1.1 Start by analyzing an overarm throw. Assume that the projectile is released with your arm at angle  $\theta$  to the vertical. Find a formula for the position vector of the projectile at time  $t$  after release, using the coordinate system shown in the figure.

1.2 Hence, calculate an expression for the time that the projectile reaches height  $h=0$ , in terms of  $\theta$

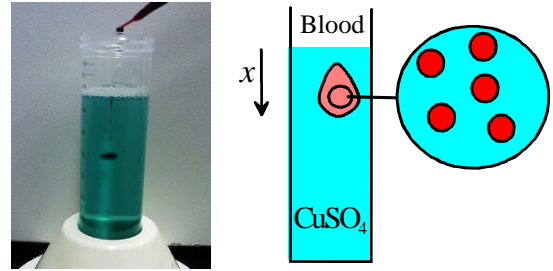
1.3 Derive an equation for the release angle that will just hit the outside of the garbage can (don't try to solve the equation by hand). Using MAPLE, find two possible release angles that satisfy the equation. (if you would like to stop MAPLE giving complex valued solutions, you can type with(RealDomain) at the start of your MAPLE file.

1.4 Derive an equation for the release angle that will just hit the inside of the garbage can, and once again, use MAPLE to find the two possible release angles satisfying the equation.

1.5 Hence, calculate  $\Delta\theta = \theta_2 - \theta_1$  for the overarm throw (you will get two ranges, of course – use the larger)

1.6 Repeat 1.1-1.5 for an underarm throw (look for a quick way to do this – you can just make a couple of small changes to your MAPLE file to get the answers you need). Which throwing style is preferable?

2. The figure illustrates a simple screening test for [anemia](#) (known as the CuSO<sub>4</sub> test – see, e.g. Rudmann “[Handbook of blood banking and transfusion medicine](#)”). The technician drops a blood specimen into a CuSO<sub>4</sub> solution with carefully controlled concentration, and observes the drop for 15sec. If the drop floats, or sinks less than a critical distance, the test indicates that the patient may have anemia. The goal of this problem is to determine the required concentration of the solution. Assume that



- A blood cell can be idealized as a sphere with radius  $R$  and mass density  $\rho$
- The CuSO<sub>4</sub> solution has mass density  $\rho_s$  and viscosity  $\eta$
- The drag force on a sphere radius  $R$  moving through a viscous fluid with speed  $v$  is  $6\pi\eta Rv$

2.1 Draw a free body diagram showing the forces acting on one of the blood cells within the blood droplet.

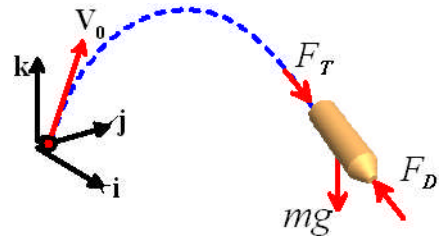
2.2 Write down Newton’s law of motion relating the depth  $x$  of the blood cell to relevant parameters. Show that  $x$  satisfies the differential equation

$$\frac{d^2x}{dt^2} + \frac{9\eta}{2\rho R^2} \frac{dx}{dt} + (\rho_s / \rho - 1)g = 0$$

2.3 Assume that a representative blood cell starts at  $x=0$  and has zero speed at time  $t=0$ . Hence, calculate expressions for the speed, and depth, of the blood cell in terms of time and other relevant parameters. Use the MAPLE ‘dsolve’ function to solve the differential equation.

2.4 Use the following values for parameters: mass density of normal blood cells  $\rho = 1.139 \text{ gram cm}^{-3}$ , mass density of CuSO<sub>4</sub> solution  $\rho_s = (18.01 + 249.7c) / (18.01 + 69.28c) \text{ gram / cm}^3$ , where  $c$  is the molar concentration of CuSO<sub>4</sub> in the solution (the number of mols of CuSO<sub>4</sub> pentahydrate divided by the number of mols of water), viscosity  $\eta = 10^{-6} \text{ N s m}^{-2}$ , radius of a blood cell  $R = 3.5 \mu\text{m}$ . Determine the concentration  $c$  that will result in blood sinking by 4cm during the 15 second test period.

3. The figure shows a rocket, which may be idealized as a particle with mass  $m$ , and which travels close to the earth's surface. It is launched from the origin with initial velocity vector  $\mathbf{V}_0 = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}$ . The projectile is subjected to the force of gravity (acting in the negative  $\mathbf{k}$  direction), a drag force



$$\mathbf{F}_D = -\frac{1}{2}\rho C_D A V \mathbf{v}$$

where  $V = \sqrt{v_x^2 + v_y^2 + v_z^2}$  is the magnitude of the rocket's velocity,  $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$  is the projectile velocity,  $C_D$  is the drag coefficient, and  $\rho$  is the air density. In addition, a thrust force  $\mathbf{F}_T = F_0\mathbf{v}/V$  is exerted on the rocket by its motor (the thrust acts in the direction of motion of the rocket).

3.1 The motion of the system will be described using the  $(x,y,z)$  coordinates of the particle. Write down the acceleration vector in terms of time derivatives of these variables.

3.2 Write down the vector equation of motion for the particle ( $\mathbf{F}=\mathbf{ma}$ ).

3.3 Show that the equation of motion can be expressed in MATLAB form as

$$\frac{d}{dt} \begin{bmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ -c_1 V v_x + c_2 v_x / V \\ -c_1 V v_y + c_2 v_y / V \\ -g - c_1 V v_z + c_2 v_z / V \end{bmatrix}$$

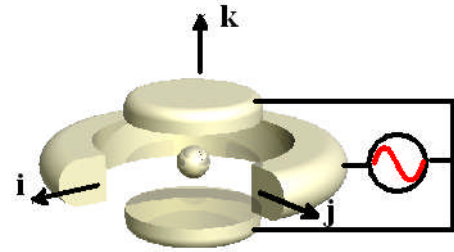
and give formulas for the constants  $c_1$  and  $c_2$  in terms of relevant parameters.

3.4 Modify the MATLAB code discussed in class (or the online notes, or the MATLAB tutorial, or last year's HW3) to calculate and plot the trajectory. Add an 'Event' function to your code that will stop the calculation when the rocket reaches the ground (THERE IS NO NEED TO SUBMIT A SOLUTION TO THIS PROBLEM).

3.5 Finally, run simulations with parameters representing a (large) model rocket ( $m=1\text{kg}$ ,  $C_D=0.1$ ,  $A=0.015\text{m}^2$ , air density  $1.02\text{ kg/m}^3$ ,  $F_0 = 20\text{N}$ ) launched in the  $(x-z)$  plane at  $0.0001\text{ m/s}$  at an angle of  $0.001$  degrees to the vertical. Plot the trajectory of the missile, and determine the time of flight, and the total horizontal distance traveled.

3.6 **Optional – for extra credit.** Suppose the rocket motor has a 5sec burn (i.e. the thrust is zero after 5sec). Find the initial launch angle that maximizes the horizontal range of the rocket, and find the corresponding range. Also, determine the maximum height reached by the rocket during its flight. **NOTE** – ode45 is not accurate enough for this problem – you should use ode113 instead. Also, set the 'Reltol' parameter to  $10^{-10}$  or better.

4. A ‘Paul Trap’ is one of several devices that use specially shaped electrostatic and electromagnetic fields to trap charged particles (or ‘ions’) (see, e.g. [Major \*et al\* \(2005\)](#) for more information). It has a number of applications: for example, as a mass spectrometer (used in chemical and biochemical analysis); or as a component in experimental quantum computers. The ions inside a particle trap are continuously moving, and follow very complicated orbits. The goal of this problem is to analyze the motion of a single charged particle inside a Paul trap.



4.1 The motion of the ion will be described by the components of its position vector  $(x,y,z)$ . Write down the velocity and acceleration of the ion in terms of time derivatives of these variables.

4.2 The ion has mass  $m$ , and is subjected to a force  $\mathbf{F} = Q\mathbf{E}$  where  $Q$  is its charge, and

$$\mathbf{E} = \frac{E_0(1 + \beta \cos \omega t)}{d} (x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k})$$

is a time dependent electric field vector. Here,  $E_0, d, \beta$  are constants that specify the magnitude and geometry of the electric and magnetic fields. Use Newton’s law to show that the components of the position vector of the ion satisfies the following equations of motion

$$\frac{d^2x}{dt^2} = \Omega^2(1 + \beta \cos(\omega t))x \quad \frac{d^2y}{dt^2} = \Omega^2(1 + \beta \cos(\omega t))y \quad \frac{d^2z}{dt^2} = -2\Omega^2(1 + \beta \cos(\omega t))z$$

where

$$\Omega = \sqrt{\frac{QE_0}{md}}$$

is a parameter (which has units of frequency).

4.3 Re-write the equations of motion as 6 first-order differential equations that can be integrated using MATLAB.

4.4 Write a MATLAB script that will solve the equations of motion to determine  $x, y, z, v_x, v_y, v_z$  as a function of time. You need not submit a solution to this problem.

4.5 An ion will be trapped if the frequency of the electric field  $\omega$  is related to  $\Omega$  correctly, and if the amplitude of the oscillation  $\beta$  lies in the correct range. Demonstrate this behavior by running computations with the following parameters (the units are arbitrary – in actual designs the frequencies are very high; typically radio frequencies).

(a)  $x = z = 0.1 \quad y = 0 \quad v_x = v_z = 0 \quad v_y = 0.1 \quad \Omega = 1 \quad \beta = 0 \quad \omega = 0$

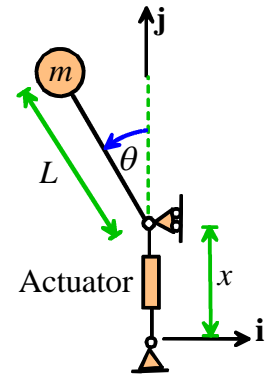
(b)  $x = z = 0.1 \quad y = 0 \quad v_x = v_z = 0 \quad v_y = 0.1 \quad \Omega = 1 \quad \beta = 30 \quad \omega = 12$

(c)  $x = z = 0.1 \quad y = 0 \quad v_x = v_z = 0 \quad v_y = 0.1 \quad \Omega = 1 \quad \beta = 30 \quad \omega = 11$

(d)  $x = z = 0.1 \quad y = 0 \quad v_x = v_z = 0 \quad v_y = 0.1 \quad \Omega = 1 \quad \beta = 30 \quad \omega = 22$

Run the tests for a time interval of 60 units. Hand in a graph showing the predicted trajectory for each case.

5. The figure shows a schematic of the parametrically excited inverse pendulum discussed on the first day of class. The pivot point moves vertically with a displacement  $x = X_0 \sin \Omega t$ . The goal of this problem is to show that at an appropriate frequency  $\Omega$  the inverted pendulum is stable.



5.1 Write down the position vector of the mass  $m$ . Hence, calculate an expression for the acceleration of the mass, in terms of  $\Omega, X_0, L$  as well as  $\theta$  and its time derivatives.

5.2 Draw a free body diagram showing the forces acting on the mass  $m$

5.3 Write down Newton's law of motion of the mass, and hence show that the angle  $\theta$  satisfies the equation

$$\frac{d^2\theta}{dt^2} - \frac{g}{L} \left(1 - \frac{\Omega^2 X_0}{g} \sin \Omega t\right) \sin \theta = 0$$

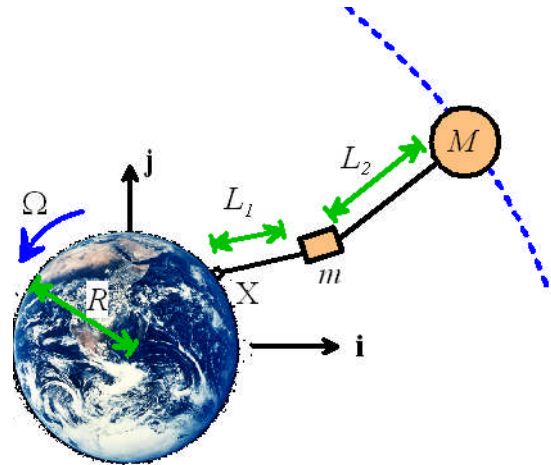
5.4 Arrange the equation of motion for  $\theta$  into a vector form that MATLAB can solve.

5.5 Write a MATLAB program to calculate and plot the angle  $\theta$  as a function of time. Show the angle in degrees. Use a relative tolerance of  $0.00001$  in the ODE solver (You don't need to submit a solution to this problem)

5.6 Plot graphs of  $\theta$  as a function of time for the following parameters. Run each simulation for 30 sec, and initial conditions  $\theta = 5 \text{ deg}$   $\dot{\theta} = 0$

- (a)  $\Omega = 60\pi$  (30cycles / sec)  $L = 0.25m$   $X_0 = 0.015m$
- (b)  $\Omega = 50\pi$  (25cycles / sec)  $L = 0.25m$   $X_0 = 0.015m$
- (c)  $\Omega = 60\pi$  (30cycles / sec)  $L = 0.5m$   $X_0 = 0.015m$
- (d)  $\Omega = 73\pi$  (36.5cycles / sec)  $L = 0.5m$   $X_0 = 0.015m$

**6. OPTIONAL: FOR EXTRA CREDIT...** The figure shows a conceptual design for a ‘space elevator’ which (if materials scientists and electrical engineers are able to achieve several technological miracles) offers a very low-cost approach to launching payloads into orbit. It consists of a satellite with mass  $M$ , which lies in the equatorial plane and is in geostationary orbit. The satellite is tethered to the earth by a cable. A ‘crawler’ with mass  $m$  rides up and down this cable transporting freight and passengers from the earth’s surface to orbit. The goal of this problem is to analyze the motion of this system. For simplicity, we will



- Assume that the system remains in the equatorial plane, so that the position of the satellite and crawler can be described by their coordinates in a fixed Cartesian basis  $\mathbf{r}_1 = x_1\mathbf{i} + y_1\mathbf{j}$ ,  $\mathbf{r}_2 = x_2\mathbf{i} + y_2\mathbf{j}$
- Neglect the mass of the cables
- Idealize the cables below and above the crawler as a combination of a spring and damper, which exert forces

$$T_1 = k_2(L_1 - a_1) + \lambda_1 \frac{d}{dt}(L_1 - a_1) \quad T_2 = k_2(L_2 - a_2) + \lambda_2 \frac{d}{dt}(L_2 - a_2)$$

to the objects attached to their ends, where  $L_1, L_2$  are the stretched lengths of the cables,  $a_1, a_2$  are the un-stretched cable lengths, and  $k_1, k_2$  and  $\lambda_1, \lambda_2$  are cable stiffnesses and damping coefficients.

6.1 The point  $X$  where the cable is attached to the surface of the earth moves as the earth rotates. Assume that at time  $t=0$  this point has position vector  $\mathbf{r}_X = R\mathbf{i}$ , where  $R$  is the radius of the earth. Write down (a) the position vector of  $X$  as a function of time, and (b) the velocity vector  $\mathbf{v}_X$  of point  $X$ , in terms of  $R$  and the angular speed of the earth  $\Omega$

6.2 Write down formulas for (a) the cable lengths; (b) unit vectors parallel to each cable; (c) unit vectors parallel to the gravitational forces acting on the crawler and satellite in terms of  $(x_1, y_1)$ ,  $(x_2, y_2)$ . Also, find expressions for  $dL_1/dt$  and  $dL_2/dt$  in terms of  $(x_1, y_1)$ ,  $(x_2, y_2)$  and their time derivatives.

6.3 Write down the acceleration vectors for the crawler and the satellite, in terms of derivatives of  $(x_1, y_1)$ ,  $(x_2, y_2)$

6.4 Draw two free body diagrams showing (a) the forces acting on the satellite, and (b) the forces acting on the crawler.

6.5 Show that the equations of motion for the crawler and satellite can be expressed as

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ v_{x1} \\ v_{y1} \\ v_{x2} \\ v_{y2} \end{bmatrix} = \begin{bmatrix} v_{x1} \\ v_{y1} \\ v_{x2} \\ v_{y2} \\ -\mu x_1 / r_1^3 + T_1(R \cos \Omega t - x_1) / mL_1 + T_2(x_2 - x_1) / mL_2 \\ -\mu y_1 / r_1^3 + T_1(R \sin \Omega t - y_1) / mL_1 + T_2(y_2 - y_1) / mL_2 \\ -\mu x_2 / r_2^3 + T_2(x_1 - x_2) / ML_2 \\ -\mu y_2 / r_2^3 + T_2(y_1 - y_2) / ML_2 \end{bmatrix}$$

where  $\mu$  is the product of the earth's mass and the gravitational constant and  $r_1 = \sqrt{x_1^2 + y_1^2}$   $r_2 = \sqrt{x_2^2 + y_2^2}$

6.6 Write a MATLAB script that will calculate the position and velocity of the satellite and crawler as a function of time. Use the following values for parameters in the problem

- The gravitational parameter  $\mu = 3.9812 \times 10^5 \text{ km}^3 \text{ s}^{-1}$
- The earth's radius  $R$  is 6472km
- The earth's angular velocity is  $\Omega = \pi / 43200 \text{ rad/s}$
- The cables have constant stiffness  $k_1 = k_2 = 1 \text{ kN} / \text{km}$  and damping  $\eta_1 = \eta_2 = 1 \text{ kNs} / \text{km}$
- The satellite has initial position and velocity  $\mathbf{r}_0 = 42241.12\mathbf{i} \text{ km}$   $\mathbf{v}_0 = 3.0719\mathbf{j} \text{ km/s}$
- The crawler has initial position and velocity  $\mathbf{r}_0 = 7015.1\mathbf{i} \text{ km}$   $\mathbf{v}_0 = 0.5102\mathbf{j} \text{ km/s}$
- As the crawler climbs the cable, the un-stretched lengths of the two cables vary with time. Take the un-stretched lengths (in km) to be

$$a_1 = \begin{cases} 637.8 + 5362.6(2\pi t / T - \sin 2\pi t / T) & t < T \\ 34332 & t > T \end{cases}$$

$$a_2 = 35863 - a_1$$

where  $T$  is the time of ascent. There is no need to submit a solution to this problem.

6.7 Plot a graph showing the trajectory (x-v-y) of the crawler and satellite as the crawler ascends the cable. Plot both trajectories on the same graph, but show them in different colors. Try the following parameters, and run all calculations for a total time of 8 days.

- $M = 5000000 \text{ kg}$   $m = 500 \text{ kg}$   $T = 5 \text{ days}$
- $M = 50000 \text{ kg}$   $m = 500 \text{ kg}$   $T = 5 \text{ days}$
- $M = 5000000 \text{ kg}$   $m = 500 \text{ kg}$   $T = 1 \text{ day}$

6.8 Plot a graphs showing the tension in each of the cables as a function of time.