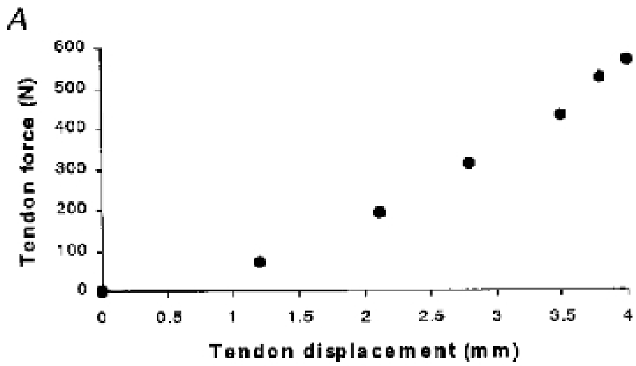
School of Engineering
Brown University

# EN40: Dynamics and Vibrations

## Homework 4: Work, Energy and Linear Momentum
## Solutions. MAX SCORE 47 points

**1.** The figure shows force-extension data for a human ankle tendon, measured in-vivo (from J Physiol. 1999 ; 521(Pt 1): 307–313. )

Estimate the total energy stored in the tendon when extended by 4mm. Compare the stored energy to the typical kinetic energy of a runner.



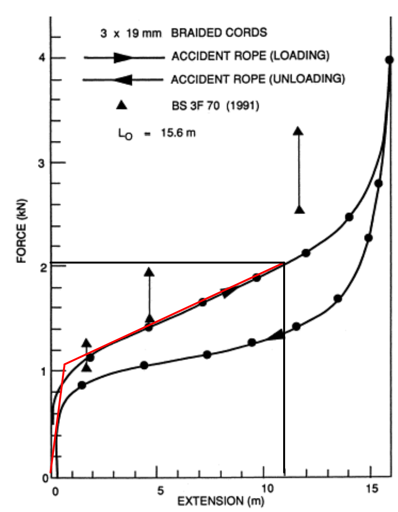

The ($x$,$y$) data can be extracted from the plot as **x** =[0,1.2,2.1,2.75,3.5,3.75,4]mm **Y**=[90,200,305,445,550,590]N**.** MATLAB can integrate this curve (use the trapz command)

```
>> x = [0,1.2,2.1,2.75,3.5,3.75,4];
>> y = [0,90,200,305,445,550,590];
>> trapz(x/1000,y)

ans =

    0.8968
```

So 0.897J is stored in the tendon. The KE of the world record womens 100m sprinter (10.49 sec), ms Griffith-Joyner weighed 59kg) is 2680J.

**[3 POINTS]**

**2.** The figure shows a force-extension curve for a bungee rope (from J. Engineering Failure Analysis, **11**, 857 (2004) ).




2.1 Estimate the energy required to stretch the cable to a tension of 2kN (use the loading curve)

The energy can be estimated from the polygon shown in the figure

$$E = \frac{1}{2}1.05\times10^3\times0.5+\frac{1}{2}10.5\times(1.05\times10^3+2\times10^3)=16275J \text{ J}$$

**[2 POINTS]**

2.2 Hence, estimate the maximum allowable weight of the jumper if the maximum force in the cable is not to exceed 2kN. Note that the cable lengt his 15.6m (see fig).

Energy is conserved during the drop. The jumper is stationary before the jump, as well as at the instant of maximum cable extension. Note that the drop height is 26m (the un-stretched length plus the extension)

$$E = mgh \Rightarrow h = 16275 / (26.6 \times 9.81) = 63.4kg$$

This is very low – which partly explains why this particular cable failed in use.

**[2 POINTS]**

**3.** One procedure for measuring human power output is to ask the test subject to jump repeatedly for a set period of time (say $T$=60 sec), count the number of jumps, and also to measure the average time $t$ that the subject is airborne during each jump (see, e.g. Eur J Appl Physiol Occup Physiol. 1983;50(2):273-82).

3.1 Use impulse-momentum to calculate the vertical speed of the subject at the instant he or she leaves the ground, in terms of the flight time $t$ and the gravitational acceleration $g$.

Energy is conserved during the jump, so the magnitude of the vertical velocity at the end of the jump must be equal to that at the start of the jump. Let $\mathbf{j}$ be a vertical unit vector. The total impulse exerted on the jumper by gravity is $-mgt\mathbf{j}$, while the change in momentum is $-mv_0\mathbf{j} - mv_0\mathbf{j}$. The impulse-momentum equation then shows that

$$2mv_0 = mgt \Rightarrow v_0 = gt / 2$$

**[2 POINTS]**

3.2 Hence, calculate the subject's kinetic energy at the instant of take-off, in terms of $g$, $t$ and the subject's mass $m$.

The kinetic energy is $mv_0^2 / 2 = m(gt)^2 / 8$

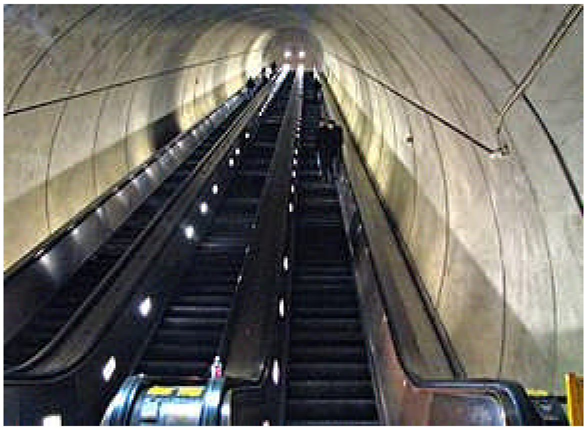
**[2 POINTS]**

3.3 Assume that the body must supply this kinetic energy for each jump. Calculate the average power developed, in terms of the test time $T$, the number of jumps $n$, as well as $g$,$t$,$m$.

The average power is the total energy expended divided by the total time of the test, i.e $P = nm(gt)^2 / 8T$

**[2 POINTS]**

**4.** The longest single-span escalator in the Western hemisphere is located at the Washington Metro station in Montgomery County, Maryland. Some technical specifications for the escalator span can be found on Wikipedia (we have not checked this data!). Additional information concerning escalator standards can be found here.



4.1 Calculate the kinetic energy of a single 80kg rider standing on the escalator

The KE is $mv^2 / 2$; we have that the speed is 27m per minute, so KE is 8.1$J$.

**[1 POINT]**

4.2 Calculate the change in potential energy of a single 80kg rider who travels the entire length of the escalator span.

The change in PE is $mgh$, where $h$ = 35m , so PE is 27468J

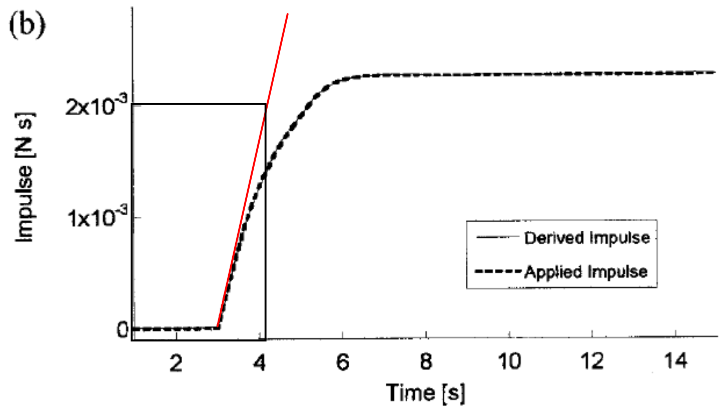
**[1 POINT]**

4.3 Assuming the escalator operates at its theoretical capacity of 9000 passengers per hour, estimate the power required to operate the escalator.

> The power is the number of passengers per second multiplied by the energy change per passenger. This gives $P$=68.7 kW

**[2 POINTS]**

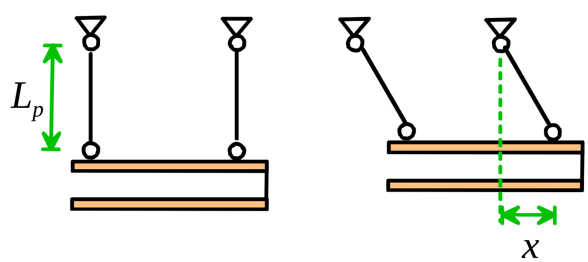
**5.** The figure shows an experimental measurement of the impulse exerted by an electrostatic thruster, as a function of time (from Rev. Sci Instr, **76**, 015105 (2005)). Estimate the maximum force exerted by the thruster.




Force is related to impulse by $F(t)=\frac{dI}{dt}$. The max force is therefore the max slope of the impulse-time curve. From the graph, this is about $1.8\times10^{-3}N$.

**[2 POINTS]**

**6.** The figure shows an experiment used by researchers at Caltech to measure the impulse caused by detonating a volume of combustible gas in a tube. The tube is mounted on a pendulum, which is at rest before the experiment starts. The gas mixture is then ignited, and the horizontal deflection $x$ at the instant when the pendulum just comes to rest is measured. Their paper states that "From elementary mechanics, the impulse is given by




$$I=m\sqrt{2gL_p\left(1-\sqrt{1-\left(\frac{x}{L_p}\right)^2}\right)}$$

where $m$ is the mass of the tube. Derive this equation.

There are two steps to this calculation: (i) The velocity of the tube just after the impulse must be calculated in terms of the displacement $x$ using energy; and (ii) the impulse can be calculated from the velocity just after the detonation, using the impulse-momentum equation.
Calculation (i) – Energy conservation, and simple geometry to calculate the height $h$ gives

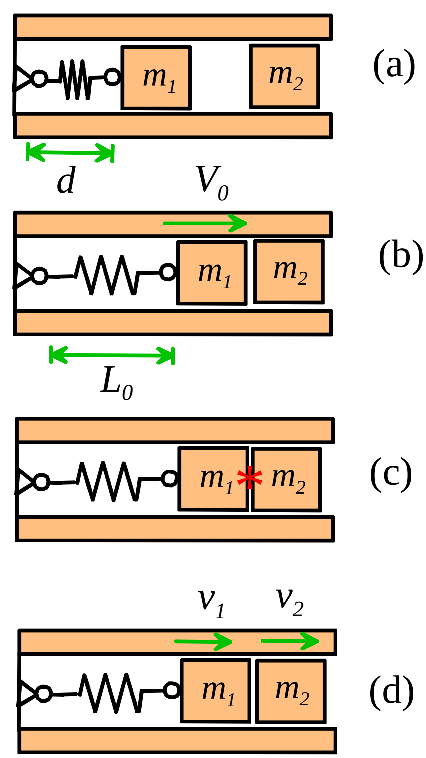
$$mv_0^2 / 2 = mgh \qquad h = L_p - \sqrt{L_p^2 - x^2} \Rightarrow v_0 = \sqrt{2gL_p\left(1 - \sqrt{1 - \left(\frac{x}{L_p}\right)^2}\right)}$$

Calculation (ii) the impulse-momentum equation $I = m(v_1 - v_0)$, and noting that the initial speed is zero, gives

$$I = m\sqrt{2gL_p\left(1 - \sqrt{1 - \left(\frac{x}{L_p}\right)^2}\right)}$$

**[5 POINTS]**

**7.** The figure shows a proposed design for a spring-loaded catapult. It operates as follows: (a) The spring is compressed to a length $d$ and then released from rest; (b) the spring returns to its unstretched length, accelerating mass $m_1$ to a speed $V_0$; (c) immediately after this point masses $m_1$ and $m_2$ collide; (d) causing mass $m_2$ to be expelled from the muzzle with speed $v_2$




The spring has stiffness $k$ and un-stretched length $L_0$, and the collision between the two masses can be characterized by a restitution coefficient $e$.

7.1 Write down the potential energy of the system in state (a).

The PE is just the energy of the spring, i.e. $V = \frac{1}{2}k(L_0 - d)^2$

**[1 POINT]**

7.2 Hence, calculate a formula for the speed of mass $m_1$ just before impact (b), in terms of $k$, $L_0$, $m_1$ and $d$.

This is a conservative system, so PE+KE is constant. At the instant just before impact, the spring returns to its unstretched length, and the KE is

$$T = \frac{1}{2}m_1 v_1^2$$

Thus

$$\frac{1}{2}k(L_0 - d)^2 = \frac{1}{2}m_1 v_1^2 \Rightarrow v_1 = \sqrt{\frac{k}{m_1}}(L_0 - d)$$

**[2 POINTS]**

7.3 Deduce expressions for the speeds $v_1$ and $v_2$ of the two masses just after the collision (d), in terms of $k$, $L_0$, $m_1$, $m_2$, $e$ and $d$.

Momentum is conserved during the impact, so $m_1\sqrt{\frac{k}{m_1}}(L_0 - d) = m_1 v_1 + m_2 v_2$

In addition, velocities before and after impact are related by the restitution coefficient

$$e\sqrt{\frac{k}{m_1}}(L_0 - d) = (v_2 - v_1)$$

These two equations can be solved for the two velocities, with the result

$$v_2 = \frac{(1+e)m_1}{m_1 + m_2}\sqrt{\frac{k}{m_1}}(L_0 - d) \qquad v_1 = \frac{(m_1 - m_2 e)}{m_1 + m_2}\sqrt{\frac{k}{m_1}}(L_0 - d)$$

**[3 POINTS]**

7.4 Show that the speed of mass $m_2$ is optimized if $m_1 = m_2$

From the preceding part,

$$v_2 = \frac{\sqrt{m_1}}{m_1 + m_2}(1+e)\sqrt{k}(L_0 - d)$$

We want to find the value of $m_1$ that maximizes this (for a fixed projectile mass $m_2$). Differentiate with respect to $m_1$ and set the derivative to zero

$$\frac{dv_2}{dm_1} = 0 \Rightarrow \frac{1}{2\sqrt{m_1}\left(m_1 + m_2\right)} - \frac{\sqrt{m_1}}{\left(m_1 + m_2\right)^2} = 0$$

$$\Rightarrow \frac{m_1 + m_2 - 2m_1}{2\sqrt{m_1}\left(m_1 + m_2\right)^2} = 0 \Rightarrow m_1 = m_2$$

**[3 POINTS]**

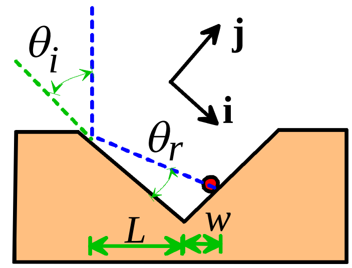
7.5 Finally, compute a formula for the energy efficiency of the optimal design.

The energy efficiency is the ratio of the KE of the projectile to the initial PE in the spring. With $m_1 = m_2$ this is

$$\frac{\frac{1}{2}m_2 v_2^2}{\frac{1}{2}k(L_0 - d)^2} = \frac{m_2 m_1}{\left(m_1 + m_2\right)^2}(1+e)^2 = \frac{\left(1+e\right)^2}{4}$$

**[2 POINTS]**

**8.** The figure shows an experiment designed to measure the restitution coefficient of nanoparticles incident on a surface (Ayesh et al, Physical Review B **81** 195422 (2010)). Bismuth nanoparticles are fired onto a V groove in a silicon wafer. The particles bounce once on impact with the sides of the groove, and then stick to the Si surface on the opposite side of the groove.




8.1 Let $\theta_i$ denote the groove angle (which is equal to the angle of incidence) and $\theta_r$ denote the angle of reflection. Neglecting friction, show that the restitution coefficient is related to $\theta_i$ and $\theta_r$ by

$$e = \frac{\tan \theta_r}{\tan \theta_i}$$

Picking **i,j** coordinates as shown in the figure, we can write the velocity vectors before and after impact as

$$\mathbf{v}_0 = V_0 \left( \cos \theta_i \mathbf{i} - \sin \theta_i \mathbf{j} \right)$$

$$\mathbf{v}_1 = V_1 (\cos \theta_r \mathbf{i} + \sin \theta_r \mathbf{j})$$

If the collision is frictionless, momentum must be conserved in the **i** direction, so that

$$V_0 \cos \theta_i = V_1 \cos \theta_r$$

The restitution coefficient formula relates the velocity components in the **j** direction before and after impact

$$V_1 \sin \theta_r = -e(-V_0 \sin \theta_i)$$

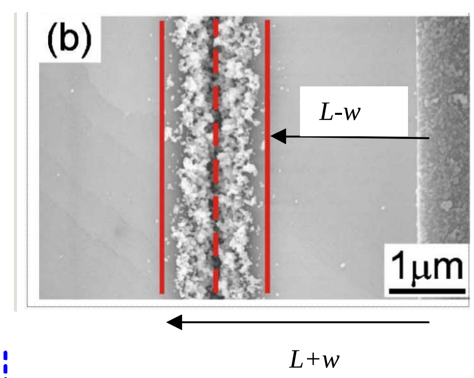
We can eliminate the velocities by dividing the second equation by the first, which gives

$$\frac{\sin \theta_r}{\cos \theta_r} = e \frac{\sin \theta_i}{\cos \theta_r}$$

This can be rearranged to give the answer stated.

**[4 POINTS]**

8.2 The figure shows an experimental result. The flat surface is the (100) plane of an Si wafer, while the groove exposes the (111) planes – this means the angle of incidence is $\theta_i = 35^0$. The dashed red line shows the center of the groove; the right hand edge can be seen as the boundary between the light and dark gray regions. Use the figure to estimate the coefficient of restitution for the particles on Si. **Neglect gravity**

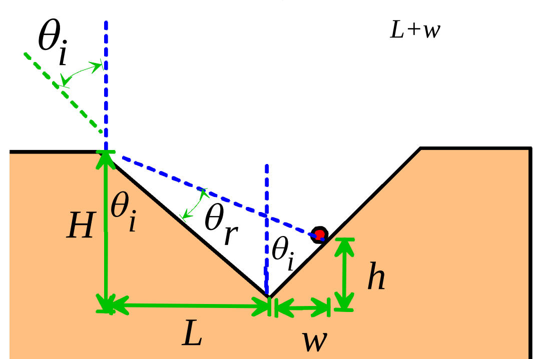



Simple geometry on the figure shown (look for right-angled triangles) indicates that

$w / h = \tan \theta_i$

$L / H = \tan \theta_i$

$(L + w) / (H - h) = \tan(\theta_r + \theta_i)$



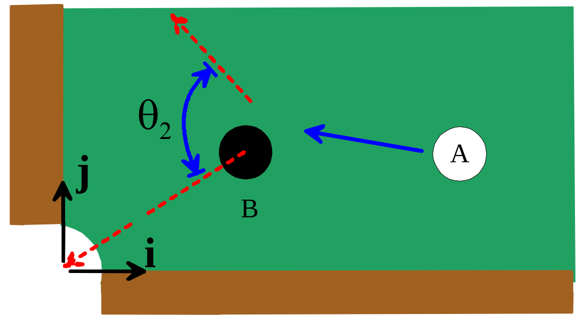

Hence $(L + w) \tan \theta_i \,/\, (L - w) = \tan(\theta_r + \theta_i)$

From the figure, $(L + w) / (L - w) \approx 1.8$, which gives

$\theta_r \approx 16.6^0$, and $e \approx 0.42$.

**[4 POINTS]**

**9.** Two balls with identical mass collide on a pool table as shown in the picture. The black ball is at rest before the collision, and the collision is frictionless with restitution coefficient *e=1*. Calculate the angle $\theta_2$ between the paths followed by the two balls after the collision (idealize the balls as particles) (*note – in general the angle depends on the direction of motion of A before collision, but e=1 is a special case.)*




Momentum conservation gives

$$m\mathbf{v}^{B1} + m\mathbf{v}^{A1} = m\mathbf{v}^{A0}$$

The restitution coefficient formula gives

$$\mathbf{v}^{B1} - \mathbf{v}^{A1} = -\mathbf{v}^{A0} - (1+e)[-\mathbf{v}^{A0} \cdot \mathbf{n}]\mathbf{n}$$

Where **n** is a unit vector parallel to the line connecting the centers of the two balls at the instant of collision.

These two equations can be solved for the velocities after collision (eg divide the first equation through by *m*, add it to the second and divide the result by two), to see that

$$\mathbf{v}^{B1} = \frac{(1+e)}{2}[\mathbf{v}^{A0} \cdot \mathbf{n}]\mathbf{n}$$

Divide the first equation through by *m* subtract the second and divide by two:

$$\mathbf{v}^{A1} = \mathbf{v}^{A0} - \frac{(1+e)}{2}[\mathbf{v}^{A0} \cdot \mathbf{n}]\mathbf{n}$$

(These equations can also be written down directly from the formulas in the notes)

The paths are parallel to the velocity vectors after collision. Recall that you can calculate the angle between the two velocity vectors from the vector formula

$$\mathbf{v}^{A1} \cdot \mathbf{v}^{B1} = \left|\mathbf{v}^{A1}\right|\left|\mathbf{v}^{B1}\right| \cos\theta_2$$

and note that

$$\mathbf{v}^{A1} \cdot \mathbf{v}^{B1} = \left(\mathbf{v}^{A0} - \frac{(1+e)}{2}[\mathbf{v}^{A0} \cdot \mathbf{n}]\mathbf{n}\right) \cdot \frac{(1+e)}{2}[\mathbf{v}^{A0} \cdot \mathbf{n}]\mathbf{n}$$

$$= \frac{(1+e)}{2}[\mathbf{v}^{A0} \cdot \mathbf{n}]^2 - \frac{(1+e)^2}{4}[\mathbf{v}^{A0} \cdot \mathbf{n}]^2 = \frac{(1+e)(1-e)}{4}[\mathbf{v}^{A0} \cdot \mathbf{n}]^2$$

Where we have noted that $\mathbf{n} \cdot \mathbf{n} = 1$ since **n** is a unit vector. Note that if *e=1*, then $\mathbf{v}^{A1} \cdot \mathbf{v}^{B1} = 0$ and therefore $\theta_2 = 90^0$.

**[5 POINTS]**