EN 4 Dynamics and Vibrations

Homework 5: Free Vibrations

1. At certain dock near Providence, the rise and fall of ocean tides occur periodically at an amplitude of 5 m and a period of 12 hours. The high tide occurs at 7 am and the low tide occurs at 1 pm. Assume the rise and fall of the tides can be approximated as a simple harmonic vibration of water level.

   1.1 Calculate the time when the water level is 2.5 m below the high water mark and the rate at which the water is falling at this moment.

   Answer:
   Write the height of the water level as:
   $$x = X \sin(\omega t + \psi)$$
   where $X = 5\,\text{m}$ and $\omega = 2\pi/12$ radian/h.

   Set t=0 at midnight. Since the high tide occurs at 7am, we have $\psi = -2\pi/3$. When the water level falls to 2.5 m below the high water mark,

   $$2.5 = 5\sin(\omega t - 2\pi/3)$$
   $$v = X\omega\cos(\omega t + \psi) < 0 \,'$$

   $$\Rightarrow t = 9 \text{ am and 9 pm.}$$

   The rate of water falling at this moment is:

   $$v = -\frac{5\sqrt{3}\pi}{6 \times 2} \text{ m/h} = -2.27 \text{ m/h} .$$

   1.2 Determine the moments when the water level is rising/falling the slowest during the day.

   Answer:
   Assuming harmonic vibration, the water level would rise/fall the slowest at high and low tides, corresponding to $t = 1$ am; 7 am; 1 pm; 7 pm.

2. A cannon barrel of mass 545 kg is observed to recoil by 1.22 m every time on firing. The recoil spring has stiffness 292 kN/m. To ensure that the barrel returns to its

original position as quickly as possible, a dashpot engaged at the end of the recoil stroke has been designed to cause critically damped recovery stroke.

2.1 Calculate the initial recoil velocity of the barrel.

Answer:

The natural frequency is

$$\omega_n = \sqrt{\frac{k}{m}} = 23.15 \ \text{rad/s}$$

From conservation of energy, we have

$$\frac{mv_0^2}{2} = \frac{kx^2}{2} \Rightarrow v_0 = \omega_n x = 28.24 \ \text{m/s}$$

2.2 Calculate the critical damping coefficient of the dashpot.

Answer:

At critical damping, the damping factor $\zeta = 1$. The critical damping coefficient is therefore:

$$c = 2m\zeta\omega_n = 2.52 \times 10^4 \ kg/s$$

2.3 Calculate the time required for the barrel to return to a position 50 mm from its initial position.

Answer:

We set the time when the barrel reaches the position x=1.22 to be t=0. At critical damping, the motion of barrel could be expressed as:

$$x(t) = (C_1 + C_2 t)e^{-\omega_n t}$$

where C_1 and C_2 are determined from the initial conditions:

$$\left.\begin{array}{l} x = 1.22 \\ v = 0 \end{array}\right\} \text{at } t\text{=0}$$

$$\Rightarrow \ C_1 = 1.22 \text{ m}; \ C_2 = 28.24 \text{ m/s}.$$

For the barrel to return to a position 50 mm from its initial position, we have:

$$0.05 = (1.22 + 28.24t)e^{-\omega_n t}$$

$$\Rightarrow \ t_1 = 0.215 \text{ s}$$

Adding the time for recoil from $x=0$ to $x=1.22$, $t_{recoil} = \dfrac{\pi}{2\omega_n} = 0.0679 \text{s}$, the total

time for the barrel return from its initial position is

$$t = t_1 + t_2 = 0.283 \text{ s}$$

3. An acrobat weighing m=60 kg walks on a tight rope between two walls. The rope is under tension and has a length of L=100 m.
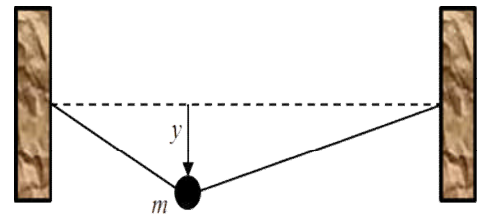
3.1 Derive the equation of motion. Show that the motion is simple harmonic vibration when the vibration amplitude is small.

Answer:

When y is small, the equation of motion for the acrobat is:

$$m\ddot{y} = -\left(\frac{T}{\alpha L} + \frac{T}{(1-\alpha)L}\right)y$$

where αL is the distance of the acrobat from the left wall. This equation can be normalized as:

$$\ddot{y} + \frac{T}{\alpha(1-\alpha)mL}y = 0$$

3.2 Suppose that the natural frequency of vibration is observed to be 10 rad/s when the acrobat is at 1/3 of the rope. Determine the tension T in the rope.

Answer:

The natural frequency is $\omega_n = \sqrt{\dfrac{T}{\alpha(1-\alpha)mL}}$.

Setting $\omega_n = 10$ rad/s when $\alpha = 1/3$, we have

$$T = \alpha(1-\alpha)mL\omega_n^2 = \frac{400000}{3} \ N$$

3.3 Determine the location of the mass where the natural frequency is the lowest. Determine the tension in the rope so that the minimum natural frequency does not

fall below 10 rad/s.

Answer:

The formula for ω_n indicates that the natural frequency is the lowest when the acrobat stands at the center of the rope. In order to meet the requirement that the lowest frequency does not fall below 10 rad/s,

$$T \geq \frac{mL\omega_{n-\min}^2}{4} = 150000 \ \text{N}$$

4. A bungee jumper weighing 90 kg ties one end of an elastic rope of unstretched length of 20 m. The stiffness of the rope k=90 N/m. The damping factor of the rope is known to be $\zeta = 0.2$.

   4.1 Write down the equation of motion and initial conditions for the motion as soon as the rope is under stretch.

Answer:

Set the position of the jumper at time *t*=0 to be *x*=0, and positive direction downwards. Then we have

$$-kx - c\frac{dx}{dt} + mg = m\frac{d^2x}{dt^2}$$

which could be simplified as

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n\frac{dx}{dt} + \omega_n^2 x - g = 0$$
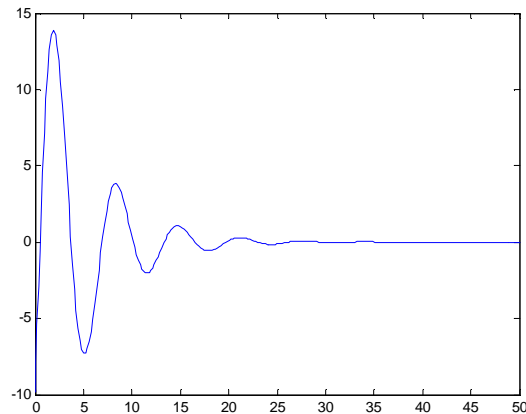
where

$$\omega_n = \sqrt{\frac{k}{m}}$$

The above equation could be recast into the standard equation of motion for vibration by introducing $y = x - g/\omega_n^2 = x - mg/k$ (with *y*=0 corresponding to the static equilibrium position),

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n\frac{dy}{dt} + \omega_n^2 y = 0$$

The initial conditions are:

$$y_0 = -mg/k = -9.8m, \quad \dot{y}_0 = \sqrt{2g \times 20} = 19.8\,\text{m/s}$$

4.2 Use MATLAB to solve the equation of motion for the first 50 seconds of the motion.

MATLAB Code:

```
function bungee_jumper
m = 90; % mass of the jumper, kg
k=90;% spring stiffness, N/m
length0 = 20; % length of the unstretched rope, m
g = 9.81; % gravitational acceleration
zeta = 0.2; % damping factor
y0=-m*g/k;
v0=sqrt(2*g*length0);
w0 = [y0,v0];
[times,sols] = ode45(@eq_of_mot,[0,50],w0);
plot(times,sols(:,1));
function dwdt = eq_of_mot(t,w)
y=w(1);v=w(2);
dwdt = [v;-2*zeta*sqrt(k/m)*v-k/m*y];
end
sols(:,1)
end
```

The above motion is one of subcritical vibration.

4.3 Suppose the motion of the jumper should decay by 99% at the 3rd cycle of motion. Determine the required damping factor of the rope and use your MATLAB code from problem 4.2 to recalculate the motion. Determine the maximum tension in the rope during the fall.
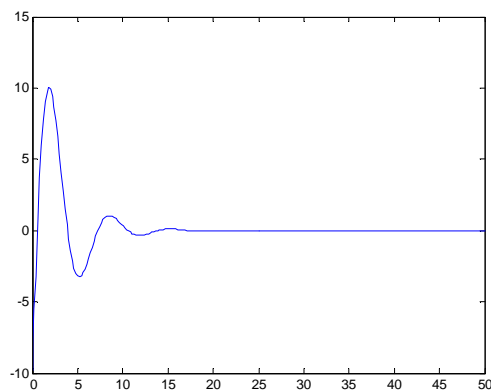
Answer:

Given that the vibration decays by 99% at the 3rd period, the logarithmic decrement is

$$\delta = \frac{1}{2}\ln\frac{1}{0.01} = 2.3$$

The damping factor is then deduced from

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = 0.34$$

The following is the recalculated motion at $\zeta = 0.34$.

The maximum stretch distance of the rope is about 20 m. Therefore, the maximum tension in the rope during the fall is

$$F_{max} = 1800\,N$$

4.4 Use your MATLAB code to search for the minimum damping factor for the rope to remain taut. What happens if the damping is too small?

Answer:

For the rope to remain taut, its length must remain larger than 20 m. This means that $y_{min} \geq -9.8\,m$. The following is a simple code used to search for the minimum damping factor.

CODE:

```
function bungee_taut
for i=1:200
    zeta =0.001*i; % damping factor increasing from 0 up to
0.2
    m = 90; % mass of the jumper, kg
    k=90;% spring stiffness, N/m
    length0 = 20; % length of the unstretched rope, m
```

```
        g = 9.81; % Gravitational accel
        time=50; %the calculation time
        y0=-m*g/k;
        v0=sqrt(2*g*length0);
        w0 = [y0,v0];
        [times,sols] = ode45(@eq_of_mot,[0,time],w0);
        taut=1; % =1 if the rope is taut and =0 if not
        [a,b]=size(sols);
        for j=1:a
            if sols(j,1)< y0
                taut=0;
            end
        end
        if taut ~=0
            break
        end
    end
end
zeta
plot(times,sols(:,1));
function dwdt = eq_of_mot(t,w)
y=w(1);v=w(2);
dwdt = [v;-2*zeta*sqrt(k/m)*v-k/m*y];
end
end
```

   The result shows that the minimum damping factor is 0.147. At this damping
factor, the calculated motion is shown below.
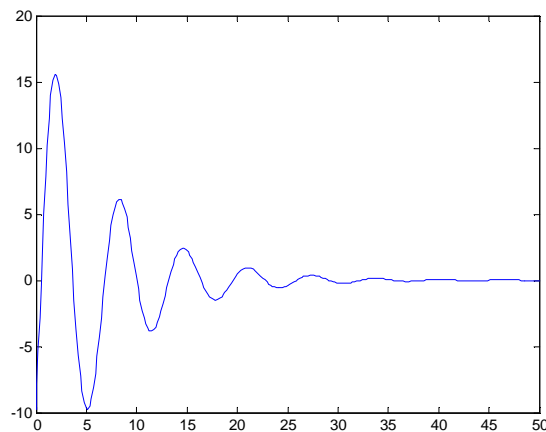
 If the damping is smaller than this critical value, the rope will loose tension and
the jumper will be in free fall during part of the motion.