School of engineering
Brown University

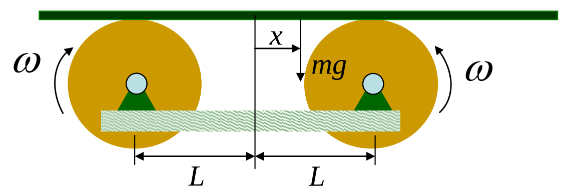
EN 4 Dynamics and Vibrations

Homework 6: Forced Vibrations

Solution

(Maximum Score 45 points + Extra Credit 8 points)

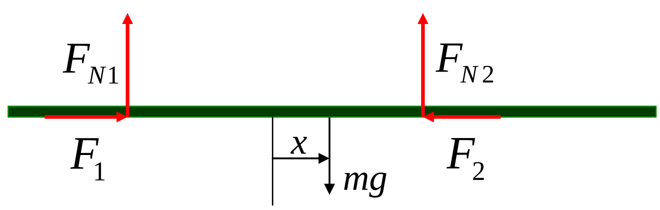
1. When a homogenous bar of mass *m* is accidentally placed on two rotating drums, as shown in the figure, it is found that the bar performs periodic oscillation. As an engineer, you have been asked to analyze this phenomenon.



Neglect the thickness of the bar in your analysis. The interface between the bar and the rotating drums has friction coefficient *f*.

1.1 Draw a free body diagram showing all the forces on the bar as its center of mass is shifted by a distance *x* toward one of the rotating drums.

Answer:



(1 point)

1.2 Determine the normal and tangential forces acting on the bar as a function of *x*.

Answer:

Balance of vertical force: $F_{N1} + F_{N2} - mg = 0$

Balance of moments about center-of-mass: $F_{N1}(L + x) - F_{N2}(L - x) = 0$

$$F_{N1} = \frac{L - x}{2L} mg\,, \quad F_{N2} = \frac{L + x}{2L} mg \qquad \text{(1 point)}$$

The tangential forces are the friction forces:

$$F_1 = f\,F_{N1} = \frac{L - x}{2L} mgf\,, \qquad F_2 = f\,F_{N2} = \frac{L + x}{2L} mgf \qquad \text{(1 point)}$$

1.3 Derive the governing equation for the horizontal motion of the bar. Determine the natural frequency of the oscillation.

Answer:

From Newton's law,

$$F_1 - F_2 = m\ddot{x}$$

Substituting F_1 and F_2 into this equation,

$$m\ddot{x} + \frac{mgf}{L}x = 0 \qquad \text{(2 points)}$$

The natural frequency is therefore

$$\omega_n = \sqrt{\frac{gf}{L}} \qquad \text{(2 points)}$$

1.4 Suppose that, at *t*=0, the bar is placed on the drums with its center of mass at *x*=0.2 m. The length of the bar is *L*=1 m and the friction coefficient is *f*=0.3. Calculate the position and velocity of the bar at *t*=1 *s*.

Answer:

Standard solution :

$$x = c_1 \cos\omega_n t + c_2 \sin\omega_n t$$

Initial condition s:

$$\dot{x}(0) = 0,\ 0 < |x(0)| \le L$$

Therefore,

$$x = x_0 \cos\omega_n t\ ,\ \ v = -x_0\omega_n \sin\omega_n t$$

Substituting the parameter values gives ($\omega_n = 1.71$ rad/s) *x*=-0.028 m and $v = -0.34$ m/s at *t*=1 s.
(3 points)

2. A quick way to estimate the damping coefficient of a machine part is to apply a harmonic force with a tunable frequency that can induce the machine part to vibrate in resonance. An inspector applied a harmonic force of 2.4 N to an airplane wing and found that the wing vibrates with amplitude of 12 cm and a period of 0.1 s at resonance. Estimate the damping coefficient of the wing.

Answer:

Under excitation by a harmonic force, the amplitude of vibration is:

$$X = \frac{F}{m\omega_n^2} \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2}}$$

At resonance $\omega = \omega_n$,

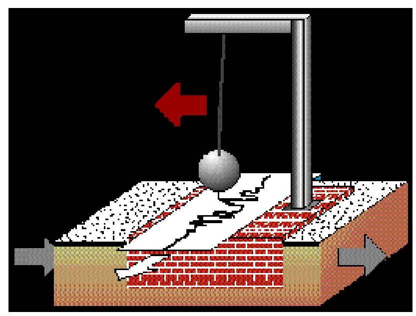
$$X = \frac{F}{2m\zeta\omega_n^2}$$

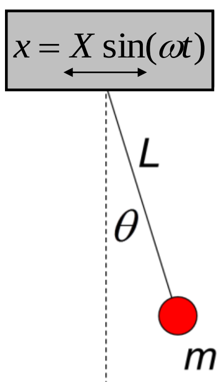
Therefore, we can calculate the damping coefficient as follows

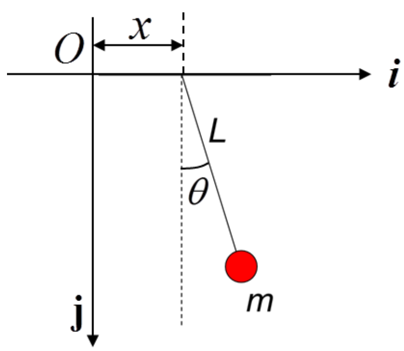
$$0.12 = \frac{2.4}{m(\omega_n)^2} \frac{1}{2\zeta} = \frac{2.4}{c\omega_n}$$

$$\Rightarrow c = \frac{1}{\pi} = 0.32 \text{ kg/s}$$ (3 points)

3. A simple seismograph consists of a pendulum with m=500 g and L=20cm attached to a rigid frame, as shown in the figure. Assume that the base is excited by a simple harmonic motion $x = X \sin \omega t$ with X=1cm.







3.1 Given the coordinates shown above, write down the position, velocity and acceleration vectors of the mass m.

Answer:

$$\mathbf{r} = (x + L\sin\theta)\mathbf{i} + L\cos\theta\mathbf{j}$$ (1 point)

$$\mathbf{v} = \dot{\mathbf{r}} = (\dot{x} + L\dot{\theta}\cos\theta)\mathbf{i} - L\dot{\theta}\sin\theta\mathbf{j}$$ (1 point)

$$\boldsymbol{a} = \ddot{\boldsymbol{r}} = \ddot{x}\boldsymbol{i} + (L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta)\boldsymbol{i} - (L\ddot{\theta}\sin\theta + L\dot{\theta}^2\cos\theta)\boldsymbol{j}$$ (1 point)

3.2 Write down the equation of motion of *m*. Do not linearize with respect to θ, except to show that the natural frequency of the pendulum is $\omega_n = \sqrt{g/L}$ in the case of small θ.

Answer:

The projection of the acceleration vector in the direction normal to the pendulum arm is

$$\begin{aligned} a_\theta &= \boldsymbol{a}\cdot\boldsymbol{e}_\theta = \ddot{x}\cos\theta + \left(L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta\right)\cos\theta + \left(L\ddot{\theta}\sin\theta + L\dot{\theta}^2\cos\theta\right)\sin\theta \\ &= \ddot{x}\cos\theta + L\ddot{\theta} \end{aligned}$$

According to Newton's law,

$$-mg\sin\theta = mL\ddot{\theta} + m\ddot{x}\cos\theta$$

or

$$\ddot{\theta} + \frac{g}{L}\sin\theta = \frac{X\omega^2}{L}\sin\omega t\cos\theta \qquad \text{(2 point)}$$

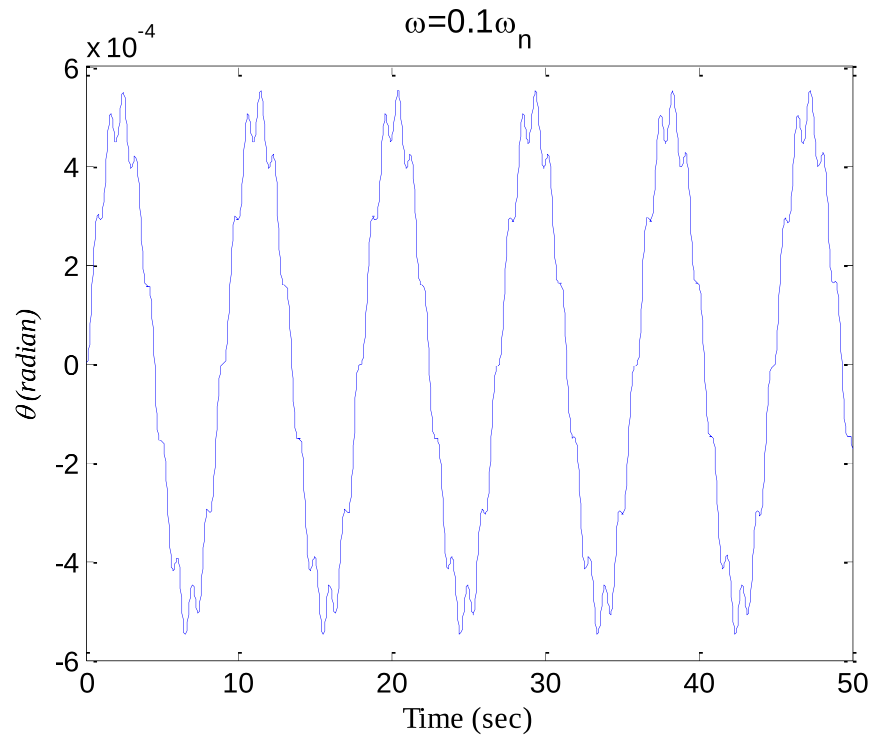
The natural frequency is $\omega_n = \sqrt{g/L}$ because the left hand side has the standard form $\ddot{\theta} + \frac{g}{L}\theta$ for small θ.

(1 point)

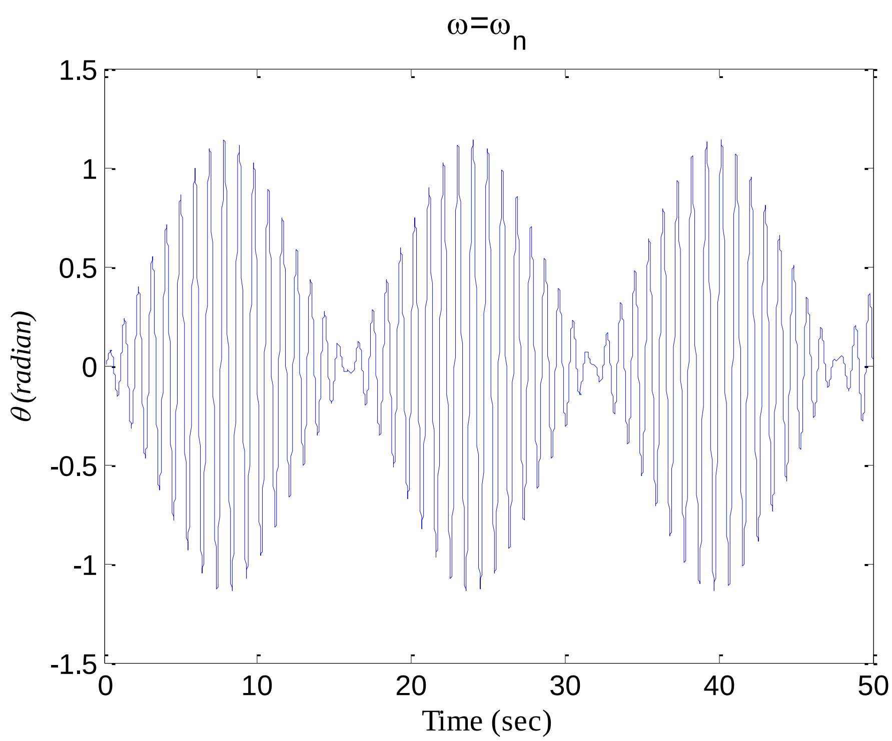
3.3 Write a MATLAB code to calculate the motion of the pendulum for $\omega = 0.1\omega_n$, ω_n, $10\omega_n$ for a time period of 0<*t*<50s. Assume the pendulum is at rest at *t*=0.

Answer:

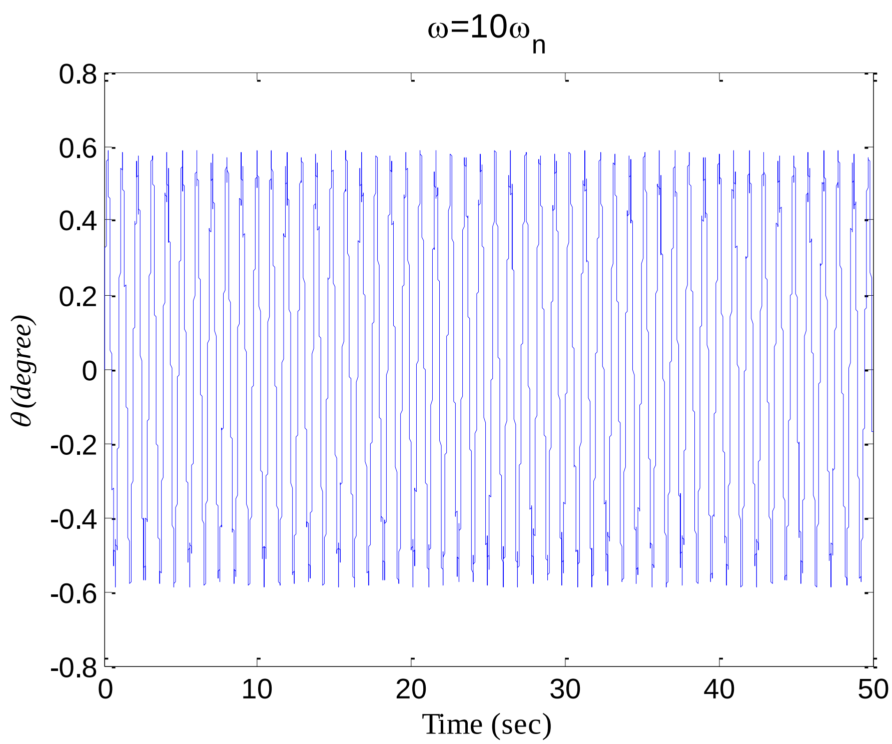
The motion of the pendulum for base frequency $\omega = 0.1\omega_n$, ω_n, $10\omega_n$ could be plotted as the following figures, separately.



ω=0.1ω_n
x 10^-4
θ (radian)
Time (sec)



ω=ω_n
θ (radian)
Time (sec)



MATLAB Code:

```
function oscillation
L = 0.2; % length of the pendulum, m
m=0.5;   % mass of the pendulum
g = 9.81; % gravitational acceleration
deltX=0.01;  % amplitude of the harmonic motion
theta0=0;
omega0=0;
w0 = [theta0,omega0];
options=odeset('RelTol',1e-6);
% Solution to omega=0.1*omega_n
figure1 =figure
axes('Parent',figure1,'FontSize',12);
box('on');
hold('all');
title({'\omega=0.1\omega_n'},'FontSize',14);
xlabel({'Time (sec)'},'FontSize',12,'FontName','Times New Roman');
ylabel({'\theta (degree)'},'FontSize',12,'FontName','Times New
Roman',...
    'FontAngle','italic');
omegaX=0.1*sqrt(g/L); %vibration frequency of the excitation, rad/s
[times,sols] = ode45(@eq_of_mot,[0,50],w0,options);
plot(times,sols(:,1));

% Solution to omega=omega_n
figure2 =figure
axes('Parent',figure2,'FontSize',12);
box('on');
hold('all');
title({'\omega=\omega_n'},'FontSize',14);
```

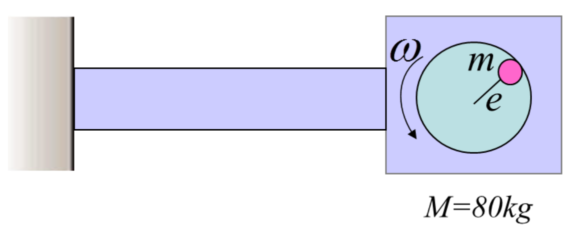
```
xlabel({'Time (sec)'},'FontSize',12,'FontName','Times New Roman');
ylabel({'\theta (degree)'},'FontSize',12,'FontName','Times New
Roman',...
    'FontAngle','italic');
omegaX=1*sqrt(g/L); %vibration frequency of the excitation, rad/s
[times1,sols1] = ode45(@eq_of_mot,[0,50],w0,options);
plot(times1,sols1(:,1));

% Solution to omega=10*omega_n
figure3 =figure
axes('Parent',figure3,'FontSize',12);
box('on');
hold('all');
title({'\omega=10\omega_n'},'FontSize',14);
ylim([-0.8 0.8]);
xlabel({'Time (sec)'},'FontSize',12,'FontName','Times New Roman');
ylabel({'\theta (degree)'},'FontSize',12,'FontName','Times New
Roman',...
    'FontAngle','italic');
omegaX=10*sqrt(g/L); %vibration frequency of the excitation, rad/s
[times2,sols2] = ode45(@eq_of_mot,[0,50],w0,options);
plot(times2,sols2(:,1));

function dwdt = eq_of_mot(t,w)
    theta=w(1);omega=w(2);
    dwdt = [omega;-
g/L*sin(theta)+deltX/L*omegaX^2*sin(omegaX*t)*cos(theta)];
end
end
```

(5 points)

4. The figure shows a tail rotor of a helicopter used to provide yaw control and torque balance. During a flight, a bird of mass *m*=500 g accidentally hit the rotor blade and got stuck on one of the blades at a distance of 15cm from the axis of rotation. This problem can be modeled as an unbalanced rotor with stiffness $k = 1\times10^5$ N/m (provided by the tail section) and an equivalent mass of *M*=80 kg (including the bird mass).



4.1 Calculate the magnitude of the vibration of the tail section of helicopter as the tail rotor rotates at 1800 rpm. Assume damping factor $\zeta = 0.01$.

Answer:

The natural frequency is:

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{10^5 \text{ N/m}}{80\text{kg}}} = 35.4 \text{ rad/s}$$

The frequency of the rotation is

$$\omega = 1800 \text{ rpm} = 188.5 \text{ rad/s}$$

$$\frac{\omega}{\omega_n} = 5.3$$

For $\zeta = 0.01$, the magnitude of oscillation of the tail rotor is

$$X = \frac{me}{M} \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}} = 0.97 \text{ mm}$$

(2 points)

4.2 Calculate the vibration amplitude at the resonance frequency.

Answer:

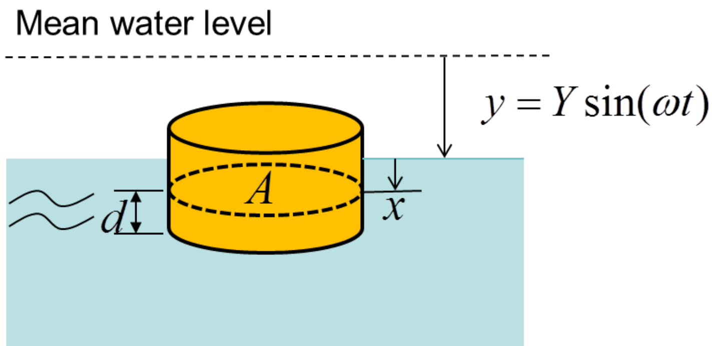
At resonance $\frac{\omega}{\omega_n} = 1$ or $\omega = 35.4 \text{ rad/s} = 338 \text{ rpm}$,

the vibration amplitude is

$$X = \frac{me}{M} \frac{1}{2\zeta} = 4.69 \text{ cm}$$

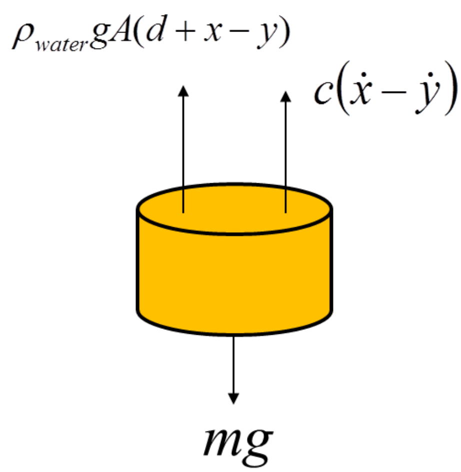
(1 point)

5. The oscillation of a ship on rough sea is modeled as a floating cylinder on water, with the water level itself also rising and falling in simple harmonic motion $y = Y \sin \omega t$. Assume the friction between the ship and water can be represented as a damper with damping coefficient c=0.4. The cross-section area of the ship is 200 m^2, the mass of the ship $m = 2 \times 10^5$ kg, and the density of the water is $\rho_{water} = 1000\ kg/m^3$. d is the draft of this ship in still water.



5.1 Draw a free body diagram showing all the forces on the ship.

Answer:



(2 points)

$\rho_{water} gAd = mg$ is the buoyancy force at static equilibrium.

5.2 Write down the equation of motion for the ship and determine the amplitude of vibration.

Answer:

Newton's law: $m\ddot{x} = -\rho_{water} gA(d + x - y) - c(\dot{x} - \dot{y}) + mg$

Setting $k = \rho_{water} gA$, this equation can be rearranged in standard form as

$$m\ddot{x}+c\dot{x}+kx=kY\sin(\omega t)+c\omega Y\cos(\omega t)$$

$$\Rightarrow \ddot{x}+\frac{c}{m}\dot{x}+\frac{k}{m}x=\sqrt{\left(\frac{kY}{m}\right)^2+\left(\frac{c\omega Y}{m}\right)^2}\sin(\omega t+\alpha)$$ (2 points)

The solution is:

$$x(t)=X\sin(\omega t-\phi)$$

$$X=\frac{Y\sqrt{1+(2\zeta\omega/\omega_n)^2}}{\sqrt{\left[1-(\omega/\omega_n)^2\right]^2+\left[2\zeta\omega/\omega_n\right]^2}}$$ (1 point)

where

$$\omega_n=\sqrt{\frac{\rho_{water}Ag}{m}}=\sqrt{\frac{g}{d}},\quad \zeta=\frac{c}{2m\omega_n}$$

5.3 Suppose the ship is designed to sustain an acceleration of $0.3g$ without running the risk of capsizing and the water level is oscillating with a period of 6s. Determine the maximum amplitude of water level oscillation that the ship can sustain.

Answer:

The natural frequency of the vibration is:

$$\omega_n=\sqrt{\frac{\rho_{water}Ag}{m}}=3.13\text{ rad/s}$$

The damping factor is

$$\zeta=\frac{c}{2m\omega_n}=\frac{0.4}{2\times 200000\times 3.13}=3.19\times 10^{-7}$$

The acceleration of the ship could be expressed as

$$a(t)=-X\omega^2\sin(\omega t-\phi)$$

where $\omega=\pi/3$ rad/s.

When the acceleration amplitude reaches $|a(t)|=0.3g$, the maximum amplitude of ship oscillation is

$$X=0.3g/\omega^2=2.68\text{ m}$$

This corresponds to the following amplitude of water level oscillation

$$Y = X\sqrt{\frac{\left[1-(\omega/\omega_n)^2\right]^2+\left[2\zeta\omega/\omega_n\right]^2}{1+(2\zeta\omega/\omega_n)^2}}$$

$$\Rightarrow Y = 2.38\text{m}$$

(3 points)

5.4 During a storm, the water level changes with amplitude $Y = 2\text{m}$ and $\omega = 2$ rad/s, determine the vibrating amplitude of the ship.

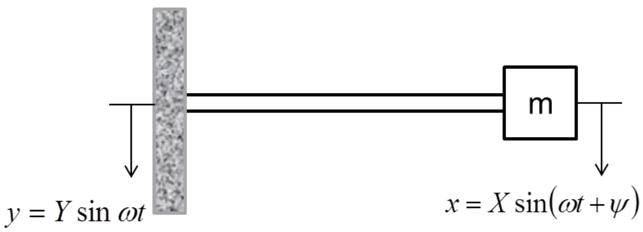
Answer:

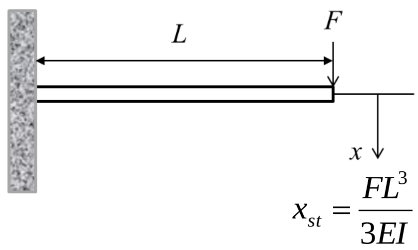
The amplitude of vibration of the ship is,

$$X = \frac{Y\sqrt{1+(2\zeta\omega/\omega_n)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2}} = 3.38\text{m}$$

(2 points)

6. The figure shows a mass m=20g attached to the end of a cantilever (whose own weight is negligible compared to m). The base of the cantilever is subject to simple harmonic motion $y = Y\sin\omega t$ and the mass is found to vibrate as $x = X\sin(\omega t+\psi)$. It is known that, under static loading, the deflection of the cantilever x can be related to an applied force F as $x_{st} = \frac{FL^3}{3EI}$ where L=10 cm is the length of the cantilever and EI is the bending stiffness of the beam.





6.1 At resonance, it is found that the mass is vibrating at amplitude $X = 2Y$ and period 0.2 s. Determine the damping coefficient and bending stiffness of the cantilever.

Answer:

Given that the mass is vibrating at amplitude $X = 2Y$ and period 0.2 s at resonance.

$$\frac{X}{Y} = 2 = \sqrt{\frac{1+\left(2\zeta\frac{\omega}{\omega_n}\right)^2}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2}} = \sqrt{\frac{1+(2\zeta)^2}{(2\zeta)^2}}$$

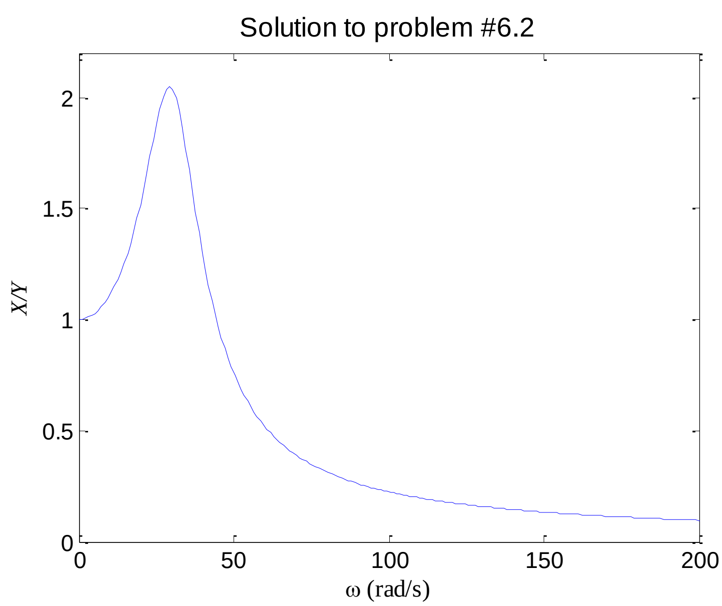
$$\Rightarrow \text{Damping factor is } \zeta = \frac{c}{2m\omega_n} = 0.289$$

where $\omega_n = 31.4$ rad/s. The damping coefficient $c = 2m\omega_n\zeta = 0.363$ kg/s (1 point)

The bending stiffness of the cantilever is

$$EI = \frac{kL^3}{3} = \frac{m\omega_n^2 L^3}{3} = 6.57\times10^{-3}\ \text{Nm}^2$$ (1 point)

6.2 Plot X/Y as a function of the base excitation frequency ω and determine the range of ω for which $X/Y \le 1$.



MATLAB Code

```
function XY_excitation
% Function to plot X/Y as a function of excitation frequency
figure1 =figure
axes('Parent',figure1,'FontSize',12);
box('on');
hold('all');
title({'Solution to problem #6.2'},'FontSize',14);
xlim([0 200]);
ylim([0 2.2]);
xlabel({'\omega (rad/s)'},'FontSize',12,'FontName','Times New Roman');
ylabel({'X/Y'},'FontSize',12,'FontName','Times New Roman',...
```

```
    'FontAngle','italic');

omegan=10*pi; % natural frequency
zeta=0.289; % damping factor
for i=1:301
    r=(i-1)/30;
    x(i)=r*omegan;
    y(i)=sqrt((1+(2*zeta*r)^2)/((1-r^2)^2+(2*zeta*r)^2));
end
    plot(x,y)
for i=2:301
    if y(i-1)>=1 && y(i)<1
        x(i)
    end
end
end
```

From the MATLAB code, we can determine the range of frequency for $X/Y<1$ is $\omega>45.03$ rad/s.

(2 points)

6.3 Assume the damping factor does not change with the length of the cantilever beam. Plot X/Y as a function of the beam length L for $\omega = 20$ rad/s. Determine the minimum length of the cantilever beam required to isolate vibration such that X/Y be kept below 0.1.

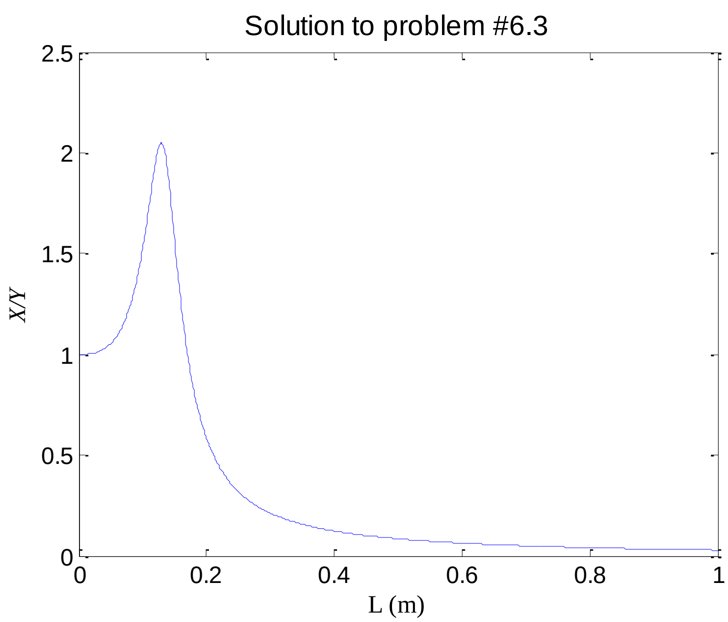
Answer:

The stiffness provided by the cantilever is expressed as a function of L

$$k = \frac{3EI}{L^3} = \frac{0.0197}{L^3}$$ (1 point)

The natural frequency is

$$\omega_n = \sqrt{\frac{k}{m}} = \frac{0.993}{L^{3/2}}$$



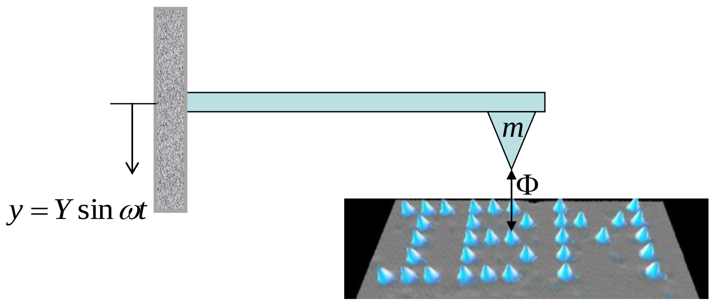
MATLAB Code

```
function XY_Length
% Function to plot X/Y as a function of beam length
figure1 =figure
axes('Parent',figure1,'FontSize',12);
box('on');
hold('all');
title({'Solution to problem #6.3'},'FontSize',14);
xlabel({'L (m)'},'FontSize',12,'FontName','Times New Roman');
ylabel({'X/Y'},'FontSize',12,'FontName','Times New Roman',...
    'FontAngle','italic');
zeta=0.289; % damping factor
EI=6.57e-3; % bending stiffness of the beam
omega=20;
m=0.02;
for i=1:1000
    L=0.001*i;
    k=(3*EI)/L^3;
    omegan=sqrt(k/m); % natural frequency
    r=omega/omegan;
    x(i)=L;
    y(i)=sqrt((1+(2*zeta*r)^2)/((1-r^2)^2+(2*zeta*r)^2));
end
    plot(x,y)
for i=2:1000
    if y(i-1)>=0.1 && y(i)<0.1
        x(i)
    end
end
end
```

From MATLAB code, the minimum length to keep X/Y below 0.1 is L=0.453 m.

(3 points)

7. Optional – For Extra Credit. The design of an Atomic force microscope (AFM) is based on a cantilever as discussed in Problem 6. In the tapping mode of AFM (http://en.wikipedia.org/wiki/Atomic_force_microscopy), the cantilever is driven to oscillate up and down at its resonance frequency. A simple cantilever is used to model an AFM, where the mass *m* represents the tip which is interacting with the surface atoms of a sample. Neglect the mass of the rest of the system in your calculation. The interactive force between the tip and sample surface obeys the following interatomic potential



$$\Phi = 4\varepsilon\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right]$$

where *r* is the distance between the AFM tip and the surface of the sample. Assume $\varepsilon = 1.6\times10^{-19}\,\mathrm{J}$ and $\sigma = 0.3\mathrm{nm}$.

7.1 Calculate the interaction force $F = \frac{\partial\Phi}{\partial r}$ and instantaneous stiffness $k_s = \frac{\partial F}{\partial r}$ between the sample and the cantilever tip. Plot F and k_s as a function of r.

Answer:

The first derivative of the potential is:

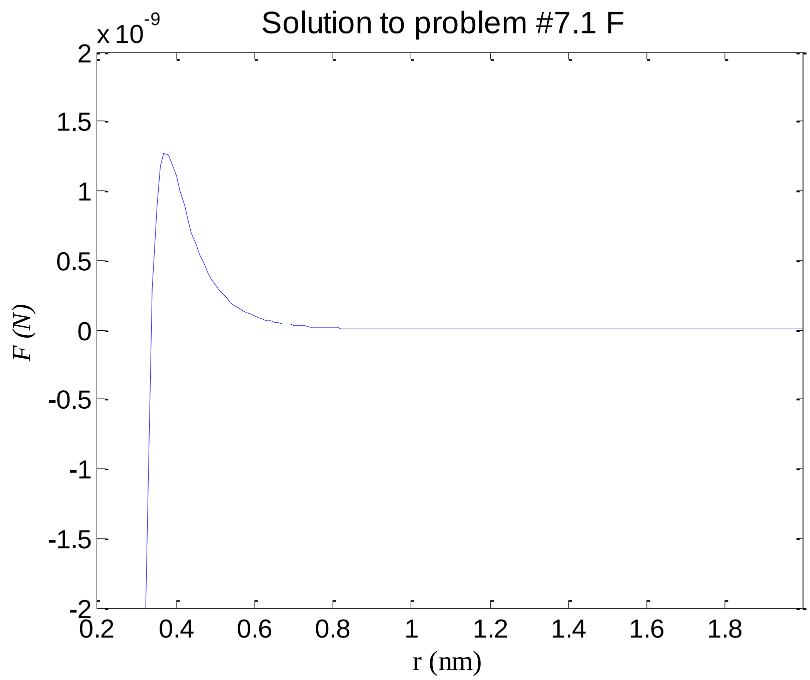
$$F = \frac{\partial\Phi}{\partial r} = 4\varepsilon\left(-\frac{12\sigma^{12}}{r^{13}} + \frac{6\sigma^{6}}{r^{7}}\right)$$

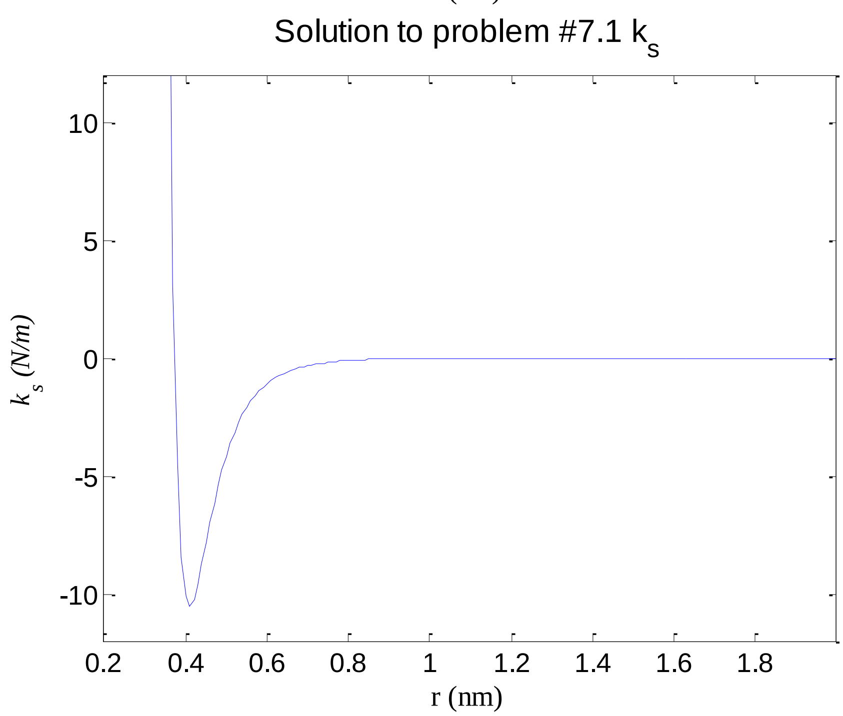
(1 Point)

the equilibrium stiffness k_s

$$k_s = \frac{\partial F}{\partial r} = 4\varepsilon\left(\frac{156\sigma^{12}}{r^{14}} - \frac{42\sigma^{6}}{r^{8}}\right)$$

(1 Point)





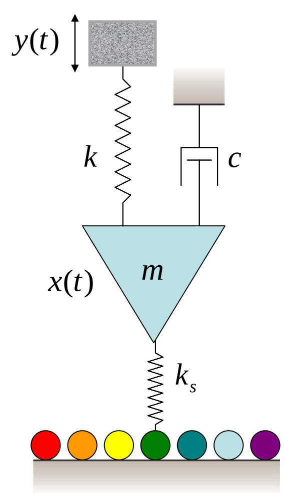
MATLAB Code

```
function potential
% Function to plot F and k_s as a function of r
sigma=0.3e-9; %the finite distance at which the inter-particle
potential is zero, nm
ems=1.6e-19; %depth of potential well, J
figure1 =figure
axes('Parent',figure1,'FontSize',12);
box('on');
hold('all');
title({'Solution to problem #7.1 F '},'FontSize',14);
```

```
xlim([0.2 2]);
ylim([-2e-9 2e-9]);
xlabel({'r (nm)'},'FontSize',12,'FontName','Times New Roman');
ylabel({'F (N)'},'FontSize',12,'FontName','Times New Roman',...
    'FontAngle','italic')
for i=1:200
    r=0.01*i*1e-9;
    x(i)=r*1e9;
    F(i)=4*ems*(6*(sigma^6)/(r^7)-12*(sigma^12)/(r^13));
end
plot(x,F)
figure2 =figure
axes('Parent',figure2,'FontSize',12);
box('on');
hold('all');
title({'Solution to problem #7.1 k_s'},'FontSize',14);
xlim([0.2 2]);
ylim([-12 12]);
xlabel({'r (nm)'},'FontSize',12,'FontName','Times New Roman');
ylabel({'k_s (N/m)'},'FontSize',12,'FontName','Times New Roman',...
    'FontAngle','italic');
for i=1:200
    r=0.01*i*1e-9;
    x(i)=r*1e9;
    k(i)=4*ems*(156*(sigma^12)/(r^14)-42*(sigma^6)/(r^8));
end
    plot(x,k)
end
```

(2 Point)

7.2 During an AFM scan, the height of the tip is tuned to an appropriate distance from the sample so that the tip vibrates in resonance. This allows the AFM to detect the surface topography of the sample as its tip scan over the entire sample surface. Write down the equation of motion for the AFM tip corresponding to the figure on the right.



Answer:

Set the initial position of this oscillation to be the equilibrium position, the dynamical equation for the displacement of the cantilever is given by

$$m\ddot{x} + c\dot{x} + (k + k_s)x = ky$$

The equation could be recast as

$$\ddot{x} + \frac{c}{m}\dot{x} + \omega_n^2 x = \frac{kY\sin(\omega t)}{m}$$

(2 points)

where

$$\omega_n^2 = \frac{k + k_s}{m}$$

7.3 At resonance, the mass vibrates at frequency 0.75×10^6 Hz due to the interaction with the sample. Given that the mass of the tip is 10^{-11} kg, and the stiffness k =200 N/m, determine the critical height of the tip when resonance occurs.

Answer:

When the resonance occurs, we have

$$\omega^2 = \frac{k + k_s}{m}$$

$$\Rightarrow k_s = 4\varepsilon\left(\frac{156\sigma^{12}}{r^{14}} - \frac{42\sigma^6}{r^8}\right) = \left(2\pi \times 10^6\right)^2 \times 10^{-11} - 200 = 22.07 \text{ N/m}$$

Determine r from MATLAB:

```
solve('156*(0.3e-9)^12/r^14-42*(0.3e-9)^6/r^8-22.07/4/(1.6e-19)','r')*1e9
```

$$\Rightarrow r = 0.357\text{nm}$$ (2 points)