



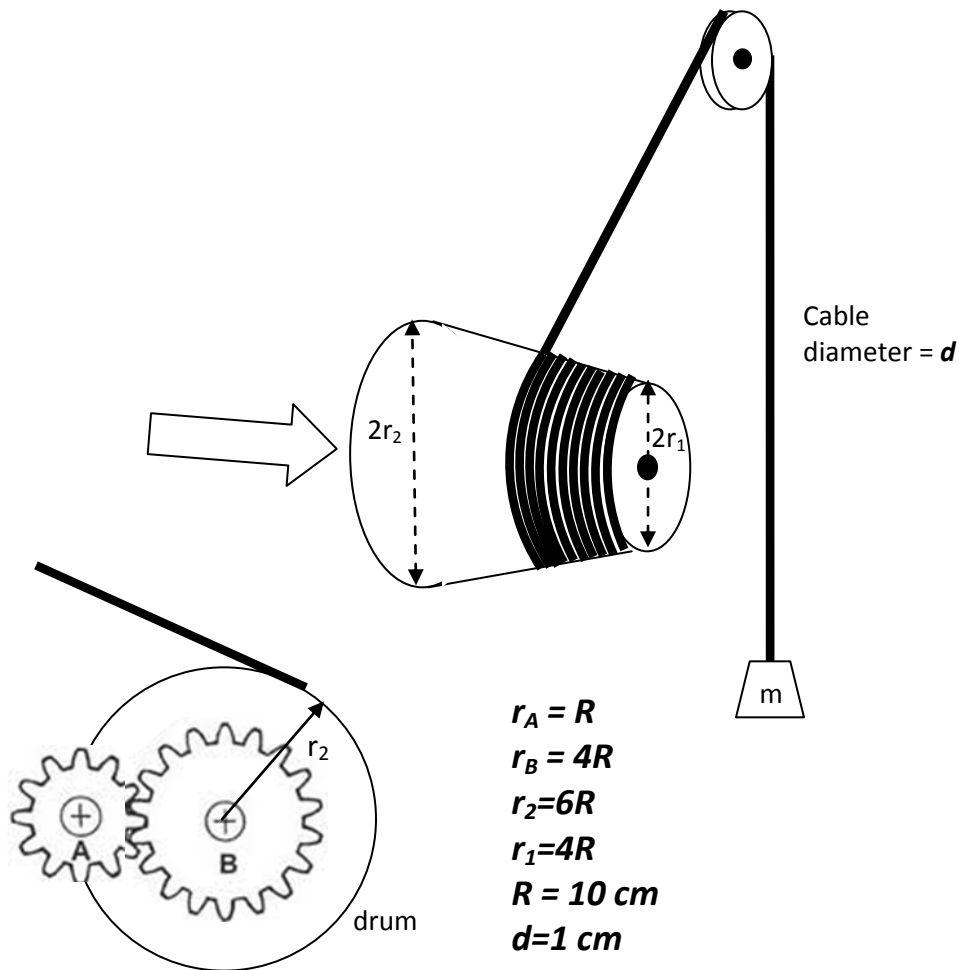
# ENGN 40 Dynamics and Vibrations

## Homework # 7

Due: Friday, April 15

1. Consider a conical hoisting drum used in the mining industry to hoist a mass up/down. A cable of diameter  $d$  has the mass connected at one end and is wound/unwound at the other end on a drum of varying radius. The drum radii at the two ends are  $r_1$  and  $r_2$  as shown. The end view of the drum is shown in the inset, which shows the gears that rotate the drum. The radius of the driver gear A is  $r_A$  and that of the driven gear B is  $r_B$ . Gear B and the conical drum are attached rigidly to the same shaft. The gear radii are as shown and the axial length of the drum is  $L = 10R$ .

Initially, the cable is completely unwound, i.e., the cable is in contact with the drum at radius  $r_1$ . Starting from rest, the driver gear rotates in counter clockwise direction at a constant angular acceleration of  $\alpha = 1 \text{ rad/s}^2$  during time  $T = 40 \text{ s}$ , followed by a constant deceleration  $-\alpha$  to rest.



(i) Derive expressions for the angular speed of gear B  $\omega_B$  as a function of time  $t$  for  $0 \leq t \leq T$  and  $T \leq t \leq 2T$  and plot  $\omega_B$  as a function of time for  $0 \leq t \leq 2T$ . Show the maximum value of  $\omega_B$  on the plot. What is the direction of rotation of the drum?

(ii) Derive expressions for the angle of rotation of the drum ( $\theta$ ) as a function of time  $t$  for  $0 \leq t \leq T$  and  $T \leq t \leq 2T$ . Plot  $\theta$  as a function of time  $t$  for  $0 \leq t \leq 2T$ . What are the values of  $\theta$  at  $t = T$  and  $t = 2T$ ?

(iii) Note that the point where the cable separates from the drum (let's call this point the "cable contact point") advances by a distance  $d$  along the drum surface for each complete revolution of the drum. Derive a relation between the radius of the drum at the cable contact point ( $r$ ) as a function of the rotation angle  $\theta$ .

(iv) Derive expressions for the speed of the mass as a function of time  $t$  for  $0 \leq t \leq T$  and  $T \leq t \leq 2T$ .

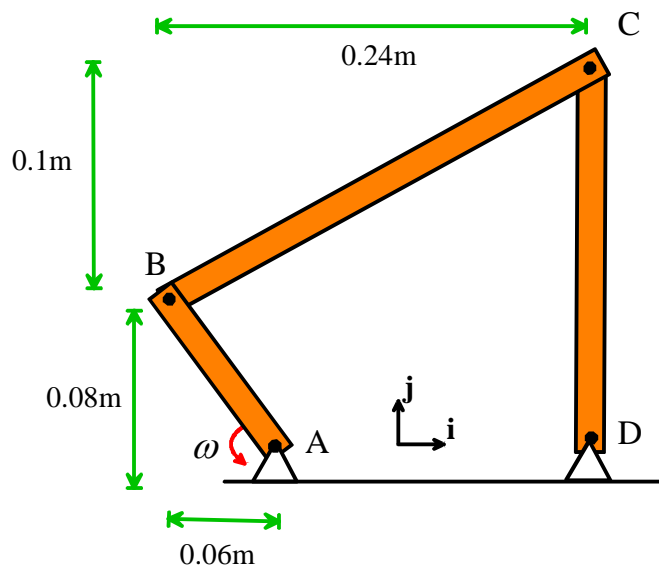
(iv) What is the total distance traveled by the mass  $m$ ?

**2.** The figure shows a crank-rocker mechanism. It consists of three rigid members AB, BC and CD connected by pin joints. Pins at A and D are attached to the ground and remain fixed. AB is the crank and CD is the rocker. As the crank AB rotates with a constant angular velocity  $\omega$ , the point C rocks back and forth along an arc. The link AB rotates at a constant angular velocity of  $\omega = 10 \mathbf{k}$  rad/s. The link CD is vertical at the instant shown.

(i) Calculate the velocity vector of point B at the instant shown in the figure, expressing your answer as components in the  $\{\mathbf{i}, \mathbf{j}\}$  coordinate system shown.

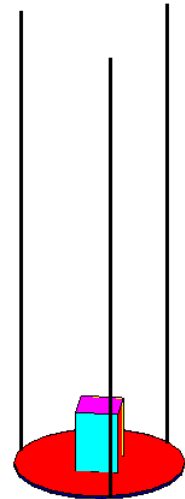
(ii) Determine the angular velocities of members BC and CD, and the velocity vector of point C.

(iii) What are the angular accelerations of BC and CD?



### 3. Trifilar pendulum analysis

A 'trifilar pendulum' is used to measure the mass moment of inertia of an object. It consists of a flat platform which is suspended by three cables. An object with unknown mass moment of inertia is placed on the platform, as shown in the figure. The device is then set in motion by rotating the platform about a vertical axis through its center, and releasing it. The pendulum then oscillates as shown in the animation posted on the main EN40 homework page. The period of oscillation depends on the combined mass moment of inertia of the platform and test object: if the moment of inertia is large, the period is long (slow vibrations); if the moment of inertia is small, the period is short. Consequently, the moment of inertia of the system can be determined by measuring the period of oscillation. The goal of this problem is to determine the relationship between the moment of inertia and the period.



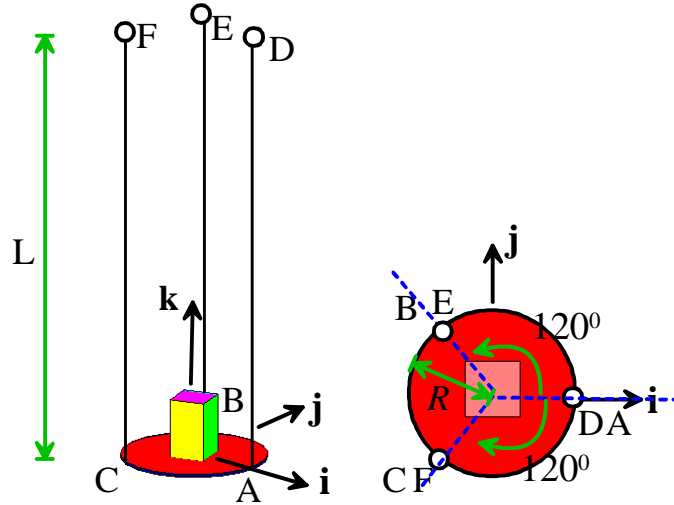
As in all 'free vibration' problems, the approach will be to derive an equation of motion for the system, and arrange it into the form

$$\frac{d^2x}{dt^2} + \omega_n^2 x = 0$$

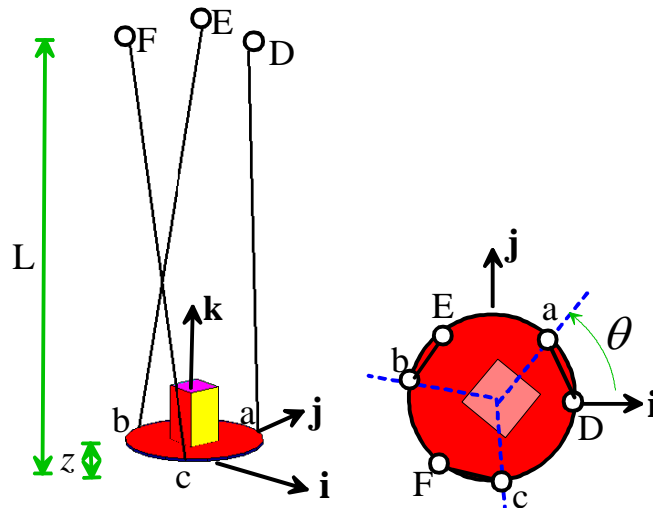
Since we are solving a rigid body problem, this equation will be derived using Newton's law  $\mathbf{F} = m\mathbf{a}_{COM}$ , and the moment-angular acceleration relation  $M\mathbf{k} = I_z\alpha\mathbf{k}$ . Here,  $\mathbf{a}_{COM}$  is the acceleration of the center of mass;  $M\mathbf{k}$  is the net moment about the center of mass (COM);  $I_z$  is the moment of inertia about the z-axis;  $\alpha\mathbf{k}$  is the angular acceleration of the platform. Note that, by symmetry, the center of mass is the center of the platform. You are already familiar with the first of these equations ( $\mathbf{F} = m\mathbf{a}$ ) for a particle. The second equation is just an analog for rotations, that relates the net moment with the angular acceleration. It will be derived in the class soon (may be on Tuesday, 4/13). But, until then, have faith in your instructors and just accept it.

Before starting this problem, watch the animation posted on the EN40 homework page closely. When you are feeling sleepy, email your Swiss bank account number to Professor Guduru. Then, notice that

- (i) The table is rotating about its center, without lateral motion
- (ii) If you look closely at the platform, you will see that it moves up and down by a very small distance. The platform is at its lowest position when the cables are vertical.



(a) The figure above shows the system in its static equilibrium position. The three cables are vertical, and all have length  $L$ . The platform has radius  $R$ . Take the origin at the center of the disk in the static equilibrium configuration, and let  $\{i, j, k\}$  be a Cartesian basis as shown in the picture. Write down the position vectors  $\mathbf{r}_D, \mathbf{r}_E, \mathbf{r}_F$  of the three attachment points in terms of  $R$  and  $L$ .



(b) Now, suppose that the platform rotates about its center through some angle  $\theta$ , and also rises by a distance  $z$ , as shown in the figure. Write down the position vectors  $\mathbf{r}_a, \mathbf{r}_b, \mathbf{r}_c$  of the three points where the cable is tied to the platform, in terms of  $R$ ,  $z$  and  $\theta$ .

(c) Assume that the cables do not stretch. Use the results of (i) and (ii) to calculate the distance between  $a$  and  $D$ , and show that  $z$  and  $\theta$  are related by the equation:

$$2R^2(1 - \cos \theta) + z(z - 2L) = 0$$

Hence, show that if the rotation angle  $\theta$  is small, then  $z \approx R^2\theta^2 / 2L$ . (Hint – use Taylor series). Since  $z$  is proportional to the square of  $\theta$ , vertical motion of the platform can be neglected if  $\theta$  is small.

(d) Write down formulas for unit vectors parallel to each of the deflected cables, in terms of  $L$ ,  $\mathbf{r}_a, \mathbf{r}_b, \mathbf{r}_c$  and  $\mathbf{r}_D, \mathbf{r}_E, \mathbf{r}_F$ . (It is not necessary to express the results in  $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$  components).

(e) Draw a free body diagram showing the forces acting on the platform and test object together.

(f) Assume that the tension has the same magnitude  $T$  in each cable. Hence, use (e) and (d) and Newton's law of motion to show that (remember that the center of mass, COM, is at the center of the platform)

$$m \left[ a_x \mathbf{i} + a_y \mathbf{j} + \frac{d^2 z}{dt^2} \mathbf{k} \right] = T \frac{\{(\mathbf{r}_D + \mathbf{r}_E + \mathbf{r}_F) - (\mathbf{r}_a + \mathbf{r}_b + \mathbf{r}_c)\}}{L} - mg \mathbf{k}$$

(g) Note that  $(\mathbf{r}_a + \mathbf{r}_b + \mathbf{r}_c) / 3$  is the average position of the three points where the cables connect to the platform. By inspection, this point must be at the center of the platform. Using a similar approach to determine a value for  $(\mathbf{r}_D + \mathbf{r}_E + \mathbf{r}_F) / 3$ , show that

$$m \left[ a_x \mathbf{i} + a_y \mathbf{j} + \frac{d^2 z}{dt^2} \mathbf{k} \right] = \{3T(1 - z/L) - mg\} \mathbf{k}$$

(h) For small  $\theta$ , we can assume  $z \approx 0$ ,  $d^2 z / dt^2 = 0$ . Hence, find a formula for the cable tension  $T$ .

(i) Finally, consider rotational motion of the system. Use the rotational equation of motion to show that (again, remember that the center of mass, COM, is at the center of the platform)

$$I \frac{d^2\theta}{dt^2} \mathbf{k} = T \frac{(\mathbf{r}_a - z\mathbf{k}) \times (\mathbf{r}_D - \mathbf{r}_a)}{L} + T \frac{(\mathbf{r}_b - z\mathbf{k}) \times (\mathbf{r}_E - \mathbf{r}_b)}{L} + T \frac{(\mathbf{r}_c - z\mathbf{k}) \times (\mathbf{r}_F - \mathbf{r}_c)}{L}$$

Either by using MAPLE to evaluate the cross products, (or if you are maple-phobic try to find a clever way to evaluate the cross products by inspection – you might like to do this as a challenge even if you love MAPLE. Then again, you may prefer to have your wisdom teeth pulled.), show that

$$I \frac{d^2\theta}{dt^2} + \frac{3R^2T}{L} \sin \theta = 0 \quad (1)$$

(j) Hence, find a formula for the frequency of small-amplitude vibration of the system, in terms of  $m, g, R, L$  and  $I$ .

(k) Write a MATLAB script that will solve equation (1) (do not linearize the equation), with initial conditions  $d\theta/dt = 0$   $\theta = \theta_0$   $t = 0$ . Use your code to compute the period of vibration for  $0 < \theta_0 < \pi/4$  and plot the results on a graph. What is the maximum allowable amplitude of vibration if the approximate formula derived in part (j) must predict the period to within 5% error? How about 1%?