



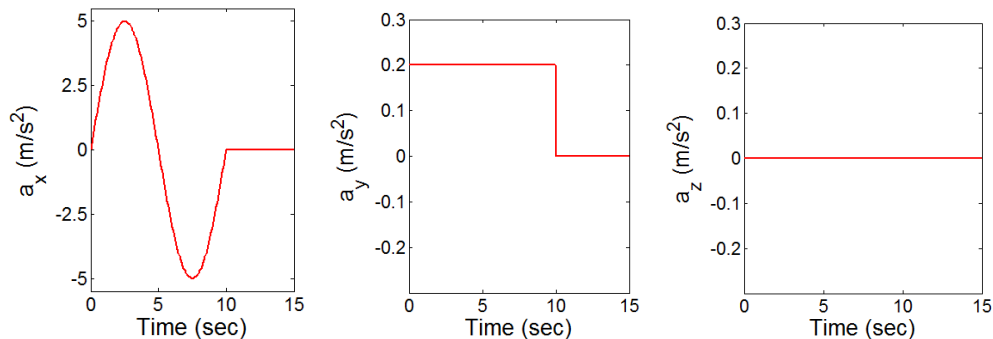
School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 2: Dynamics of Particles

Due Friday Feb 10, 2012

55 POINTS + 19 EXTRA CREDIT



1. A [three-axis accelerometer](#) mounted on an [inertial platform](#) with fixed orientation measures three components of acceleration shown in the figure (the first figure can be approximated as a sinusoidal function). If the platform is initially at rest at the origin, what is its position vector after 15sec?

This is just a straightforward but tedious exercise in integrating the signals. From the graphs:

$$a_x = \begin{cases} 5 \sin(2\pi t / 10) & t < 10 \\ 0 & t > 10 \end{cases} \Rightarrow v_x = \begin{cases} \int_0^t 5 \sin(2\pi t / 10) = \frac{25}{\pi} [1 - \cos(2\pi t / 10)] & t < 10 \\ \int_0^{10} 5 \sin(2\pi t / 10) = 0 & t > 10 \end{cases}$$

$$\Rightarrow x = \begin{cases} \int_0^t \frac{25}{\pi} [1 - \cos(2\pi t / 10)] = \frac{25}{\pi} t - (25 \times 5 / \pi^2) \sin(2\pi t / 10) & t < 10 \\ \int_0^{10} \frac{25}{\pi} [1 - \cos(2\pi t / 10)] dt = \frac{250}{\pi} & t > 10 \end{cases}$$

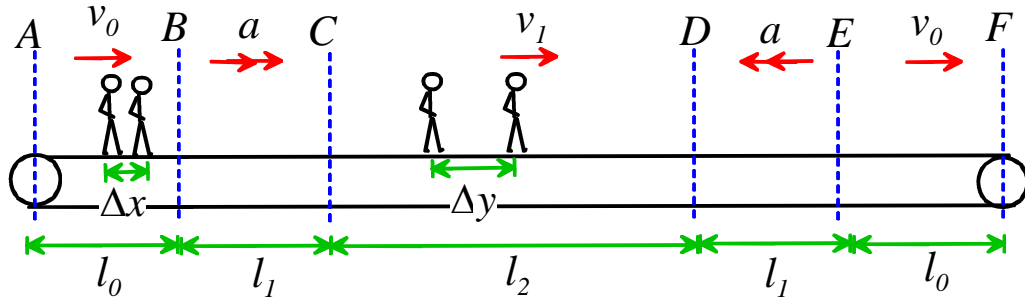
[1 POINT FOR ACCELERATION, 2 POINTS FOR CALCULATING x CORRECTLY]

$$a_y = \begin{cases} 0.2 & t < 10 \\ 0 & t > 10 \end{cases} \Rightarrow v_y = \begin{cases} \int_0^t 0.2 dt = 0.2t & t < 10 \\ \int_0^{10} 0.2 dt = 2 & t > 10 \end{cases} \Rightarrow x = \begin{cases} \int_0^t 0.2t dt = 0.1t^2 & t < 10 \\ \int_0^{10} 0.2t dt + 2(t-10) = 10 + 2(t-10) & t > 10 \end{cases}$$

(You can do these integrals with Mathematica if you prefer)

At time $t=15$ s, then, the position vector is $\mathbf{r} = \frac{250}{\pi} \mathbf{i} + 20 \mathbf{j} \quad m$

[2 POINTS]



2. The figure shows a design for a high-speed moving walkway (see http://www.jfe-steel.co.jp/archives/en/nkk_giho/84/pdf/84_10.pdf for a detailed description of this general type of design, or <http://www.youtube.com/watch?v=uwHer1RrYg8> for a movie of such a walkway in action). A passenger standing on the walkway passes through five regions:

- (i) between A and B she moves at constant speed v_0 ;
- (ii) between B and C she accelerates with constant acceleration a ;
- (iii) between C and D she moves with constant (high) speed v_1 ; and
- (iv) between D and E she decelerates at constant rate $-a$, and
- (v) between E and F she travels at speed v_0 again.

In this problem, you will work through a few of the calculations required to design the walkway.

2.1 Suppose that a passenger arrives at point B at time $t=0$ and travels between B and C. Write down formulas for her speed $v(t)$ and distance traveled $y(t)$ in terms of v_0 and a .

The constant acceleration straight-line motion formulas give

$$v(t) = v_0 + at \quad y(t) = v_0 t + at^2 / 2$$

[2 POINTS]

2.2 Noting that the passenger must acceleration from speed v_0 , to speed v_1 at constant acceleration between B and C, find an expression for the length l_1 of the walkway between B and C, in terms of v_0 , v_1 and a .

The time taken to travel from B to C can be calculated by solving

$$l_1 = v_0 t + at^2 / 2 \Rightarrow t = -\frac{v_0}{a} + \sqrt{\left(\frac{v_0^2}{a^2} + \frac{2l_1}{a}\right)}$$

We know the speed at B, so

$$v_1 = v_0 + at = \sqrt{\left(v_0^2 + 2al_1\right)}$$

$$\Rightarrow l_1 = (v_1^2 - v_0^2) / 2a$$

If you happen to remember it, you can also use the formula $v_1^2 = v_0^2 + 2al_1$ to get the solution directly.

[2 POINTS]

2.4 Find a formula for the time taken for a passenger to travel from A to E, in terms of l_0, l_2, v_0, v_1, a

The total time to travel is

$$t = 2l_0 / v_0 + l_2 / v_1 + 2(v_1 - v_0) / a$$

[2POINTS]

2.6 The walkway in Toronto has $v_1 = 400 \text{ ft} / \text{min}$ $v_0 = 125 \text{ ft} / \text{min}$. Its total length is 912ft (http://www.thyssenkruppelevator.com/downloads/Executive_Manual_Testimonials.pdf). In this design $l_0 = 0$ Use the movie posted online to estimate the acceleration a (you can time how long it takes to transition from on velocity to the other), and hence determine l_1 . Calculate the total time of travel along the walkway (assume the passenger stands still on the walkway), and calculate how much time is saved per passenger by using a high-speed walkway compared to a low speed (125 ft/min) version.

It takes about 7-10 seconds to slow down at the end of the walkway, so $a \approx 0.6 \text{ ft} / \text{s}^2$

The transition length follows as $(v_1^2 - v_0^2) / 2a \approx 35 \text{ ft}$. The central portion has length 842 ft

The total time of travel is $t = 2l_0 / v_0 + l_2 / v_1 + 2(v_1 - v_0) / a = 140 \text{ sec}$.

The time of travel at 125ft/min is 438 sec. About 300sec (5 mins) is saved per passenger.

[3 POINTS]

2.3 **OPTIONAL (hard)** Suppose that a row of passengers stand on the walkway. They are a small distance Δx apart while traveling from A to B. How far apart are they when traveling between C and D? Express your answer in terms of Δx , v_1 and v_0 .

There are two ways to solve this problem – a straightforward but tedious way, and a direct but devious way.

To do the the straightforward but tedious calculation, note that the leading passenger accelerates for a time $\Delta t = \Delta x / v_0$ before the second passenger reaches B. Suppose that the second passenger travels for a time t after passing B. The two passengers travel distances

$$y_1 = v_0(t + \Delta t) + a(t + \Delta t)^2 / 2$$

$$y_2 = v_0 t + at^2 / 2$$

The distance between them follows as

$$y_1 - y_2 = v_0 \Delta t + a(t + \Delta t)^2 / 2 - at^2 / 2 \approx v_0 \Delta t + at \Delta t$$

Here, we have assumed $a \Delta t^2 / 2$ is very small and can be neglected compared to the other terms.

Next, calculate the time taken for the passengers to travel between B and C – we know the speeds at B and C, and know they must be related by the constant acceleration straight-line motion formula, so

$$v_1 = v_0 + at \Rightarrow t = (v_1 - v_0) / a$$

Substituting this into the preceding formula shows that the separation between passengers at B is

$$y_1 - y_2 = v_0 \Delta t + (v_1 - v_0) \Delta t = v_1 \Delta t = v_1 \Delta x / v_0$$

The direct but devious solution is to note that the number of passengers per second passing any fixed point on the walkway must be constant (otherwise passengers would accumulate or disappear somewhere in the middle.). The number of passengers passing a fixed point between A and B per second is $n = v_0 / \Delta x$. Now suppose that the same row of passengers pass through the accelerating region and then travel along between B and C. If they are now spaced by a distance Δy , the number of passengers passing a fixed point per second is $n = v_1 / \Delta y$. Since n remains constant,

$$v_0 / \Delta x = v_1 / \Delta y \Rightarrow \Delta y = v_1 \Delta x / v_0$$

[4 POINTS]

2.5 **OPTIONAL** (also hard) Suppose that a passenger walks along the walkway at constant speed w relative to the travelling walkway between B and C. What is her acceleration? Express your answer as a function of her distance y from B, as well as other relevant parameters.

Her acceleration can be calculated as

$$a = \frac{dv}{dt} = \frac{dv}{dy} \frac{dy}{dt}$$

where v is the velocity of the point on the walkway that is instantaneously under the passenger (from the chain rule). We can calculate the velocity of the walkway as a function of distance y from B using the constant acceleration straight line motion formula

$$v = \sqrt{v_0^2 + 2ay}$$

Note also that the passenger walks at speed w relative to the walkway, her speed with respect to B is

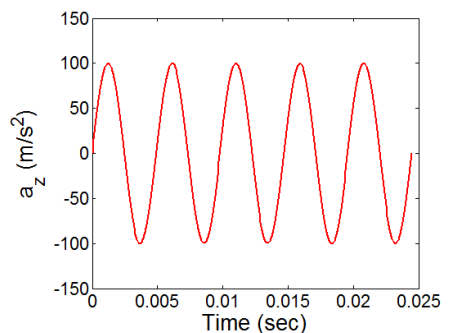
$$\frac{dy}{dt} = v + w$$

Her acceleration follows as

$$a = \frac{a}{\sqrt{v_0^2 + 2ay}} \left(w + \sqrt{v_0^2 + 2ay} \right) = a \left(1 + \frac{w}{\sqrt{v_0^2 + 2ay}} \right)$$

[4 POINTS]

3. In an experiment to measure the stiffness of a thin, extensible wire, an accelerometer with mass 100 grams is suspended from the end of the wire. (Recall that the stiffness of a wire (k) relates the force (F) to the extension x of the wire by $F=kx$). The accelerometer is then tapped lightly to start it vibrating. The figure shows the measured vibration (note the simple harmonic motion).



3.1 Determine the period, angular frequency, and frequency of the acceleration

There are five cycles of vibration in about 0.0245 seconds. The period is the time for one complete cycle, i.e. $T \approx 0.005s$. The angular frequency follows as $\omega = 2\pi / T = 400\pi$, and the frequency is $1/T=200$ cycles/s (or Hertz).

[2 POINTS]

3.2 Hence, determine the amplitude of the displacement of the accelerometer

For simple harmonic motion the displacement amplitude is related to the acceleration amplitude by $\Delta X = \Delta A / \omega^2$. From the figure, $\Delta A = 100m / s^2$, so the displacement is $100 / (400\pi)^2 = 63.3\mu m$ (one μm is $10^{-6}m$).

[1 POINT]

3.2 By considering the force necessary to cause the measured acceleration, and using 3.2 to determine the corresponding stretch of the wire, determine the stiffness of the wire. Express your answer in N/m .

The force exerted by the wire on the accelerometer can be determined from Newton's law:

$$F_a(t) = m\Delta A \sin \omega t$$

The displacement of the accelerometer is $x(t) = -(\Delta A / \omega^2) \sin \omega t$ (check this by differentiating x twice wrt time).

The force exerted by the accelerometer on the wire is equal and opposite to the force exerted by the wire on the accelerometer: $F_w = -F_a$. Finally substituting into $F_w(t) = kx(t)$

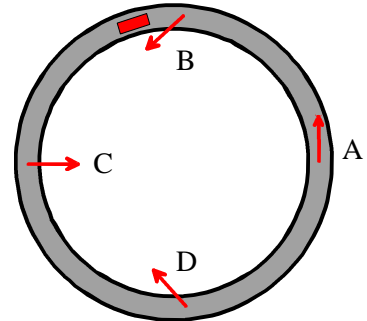
$$-m\Delta A \sin \omega t = k[-(\Delta A / \omega^2) \sin \omega t]$$

$$\Rightarrow k = m\omega^2$$

Hence $k = 0.1.(400\pi)^2 = 1.6 \times 10^5 N / m$

[3 POINTS]

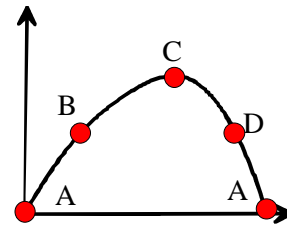
4. The figure shows the direction of the acceleration vector for a vehicle traveling counterclockwise around a circular track, at four different points along the path. The vehicle starts at A. Sketch a graph showing the variation of the car's speed (i.e. the magnitude of the velocity) with distance traveled (label points A,B,C,D on your graph).



The best way to think about this problem is to recall that the acceleration of a particle can be expressed in normal-tangential components as

$$\mathbf{a} = \frac{dV}{dt} \mathbf{t} + \frac{V^2}{R} \mathbf{n}$$

Where \mathbf{t} is tangent to the path, and \mathbf{n} is normal to the path, towards the center of curvature. The tangential component of velocity tells us about the slope of the speed, and the normal component tells us about its value. Thus, at A, the speed must be zero, and its slope must be positive. At B, the speed must be greater than zero, and must be increasing. At C there is no tangential component, so $dV/dt=0$. The speed must be a maximum. At D, the speed must be decreasing. So the graph must look something like this

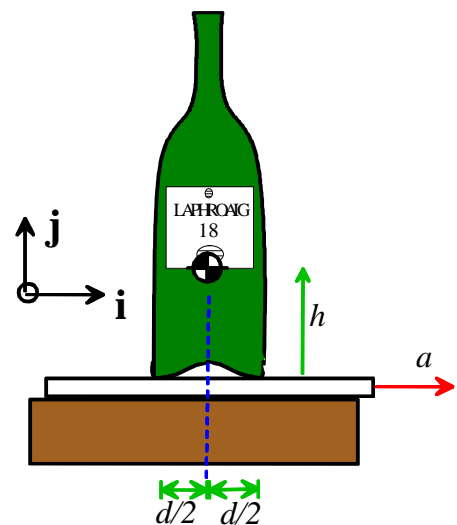
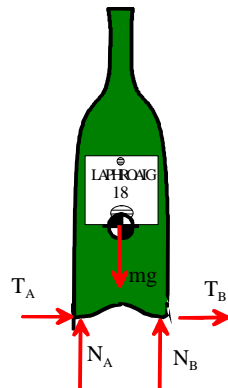


[4 POINTS]

4. The figure shows the ‘tablecloth’ trick demonstrated in class. The bottle has diameter d at the base, and its center of mass is a height h above the table. The coefficient of friction between cloth and bottle is μ . The cloth is pulled horizontally with an acceleration $a > \mu g$ so the cloth slips under the bottle.

4.1 Draw a free body diagram showing the forces acting on the bottle.

[2 POINTS]



4.2 Assuming that the bottle does not tip, calculate its horizontal acceleration.

$$\text{Newton's law gives } (T_A + T_B)\mathbf{i} + (N_A + N_B - mg)\mathbf{j} = ma_{\text{Bottle}}\mathbf{i}$$

$$\text{The cloth slips, so } T_A = \mu N_A \quad T_B = \mu N_B$$

$$\text{Hence } N_A + N_B = mg \quad (T_A + T_B) = ma_{\text{Bottle}} = \mu(N_A + N_B) = \mu mg$$

$$\text{The acceleration is therefore } a_{\text{Bottle}} = \mu g$$

[2 POINTS]

4.3 Show that the bottle will tip over if h/d exceeds a critical value, and give an expression for the maximum allowable value of h/d for the trick to work.

Moments about the COM gives

$$(T_A + T_B)h + (N_B - N_A)d / 2 = 0$$

$$\Rightarrow 2\mu mgh / d + (N_B - N_A) = 0$$

Recall that $N_A + N_B = mg$ and solve for N_A, N_B :

$$N_B = mg / 2 - \mu mgh / d \quad N_A = mg / 2 + \mu mgh / d$$

Notice that $N_B = 0$ if $h / d > 1 / 2\mu$ which means the bottle will tip. So make sure that $h / d < 1 / 2\mu$

[3 POINTS]

6. An unbalanced rotor that is spun at constant speed by a motor attached to its hub can be idealized as a particle with mass m located at the center of mass of the rotor, which is a distance L from the hub as shown in the figure.

6.1 Write down the position vector of the particle (i.e. center of mass) in terms of L and the angle θ . Hence, derive expressions for its acceleration in terms of θ , $\omega = d\theta / dt$ and L . Use the basis shown, and assume that ω is constant.

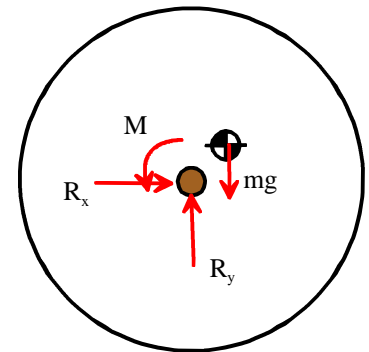
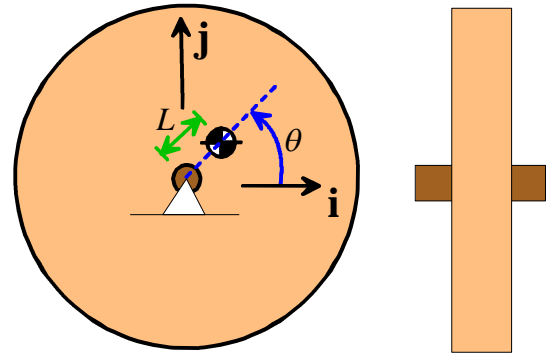
$$\mathbf{r} = L \cos \theta \mathbf{i} + L \sin \theta \mathbf{j}$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = L\omega(-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) \quad [3 \text{ POINTS}]$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -L\omega^2(\cos \theta \mathbf{i} + \sin \theta \mathbf{j})$$

6.2 Draw the forces and moments acting on the rotor
Gravity should be included.

[3 POINTS]



6.3 Hence, calculate expressions for the horizontal and vertical reaction forces acting at the rotor hub as functions of time.

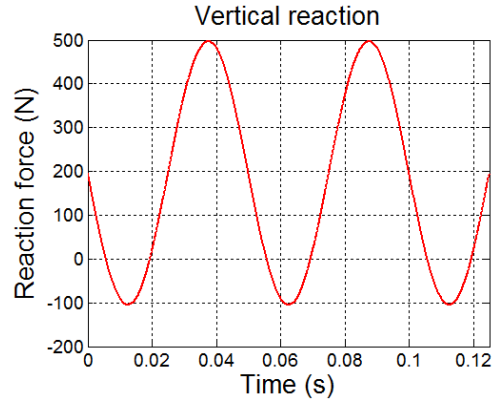
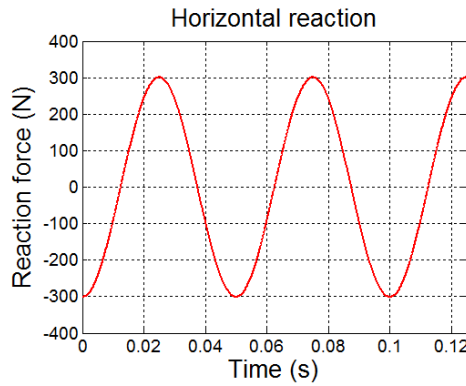
Newton's law gives

$$R_x \mathbf{i} + (R_y - mg) \mathbf{j} = -mL\omega^2(\cos \theta \mathbf{i} + \sin \theta \mathbf{j})$$

[3 POINTS]

$$\Rightarrow R_x = -Lm\omega^2 \cos \omega t \quad R_y = mg - Lm\omega^2 \sin \omega t$$

6.4 The graphs show reaction forces measured experimentally. Determine the mass of the rotor, and the distance of the center of mass from the axis of rotation. Use SI units.

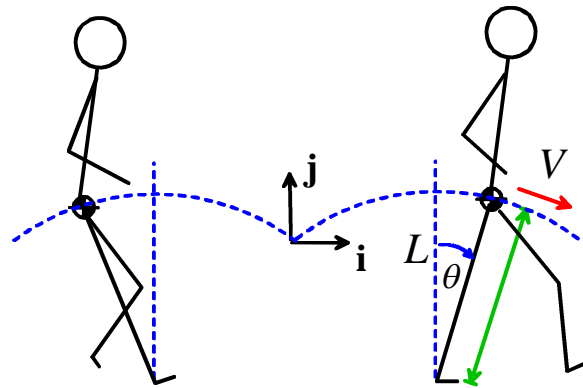


The mean value of R_y is 200N and (from 2.3) must equal mg . Therefore $m = 200 / 9.81 = 20.39kg$
 Note that R_x and R_y are both harmonic (as predicted in 2.3), with amplitude 300N. The period is 0.05sec.
 Therefore $\omega = 2\pi / T = 40\pi \text{ rad / s}$.

From 2.3 we see that $Lm\omega^2 = 300N \Rightarrow L = 300 / (m\omega^2) = 0.93 \times 10^{-3} m \text{ (0.93mm)}$ [5 POINTS]

7. In this problem you will estimate the maximum possible speed of a human walker. As you probably know, the difference between running and walking is that, when you run, your feet leave the ground, whereas they remain in contact with the ground if you walk. Your goal is to estimate the speed at which your feet lose contact with the ground.

5.1 Suppose that you walk with your load bearing leg straight – your center of mass then travels along a series of circular arcs (with each arc corresponding to single step), as shown in the figure. Assume that your COM moves with constant (absolute) speed V . Find formulas for the horizontal and vertical components of acceleration of the COM as functions of the angle θ as well as V and L .



Note that since the speed of the COM is constant, the magnitude of the acceleration is V^2 / L and the direction is $-\sin \theta \mathbf{i} - \cos \theta \mathbf{j}$

$$\mathbf{a} = -\frac{V^2}{L}(\sin \theta \mathbf{i} + \cos \theta \mathbf{j})$$

[2 POINTS]

5.2 Use Newton's laws of motion to calculate formulas for the reaction forces acting on the foot in contact with the ground

The figure shows a free body diagram. Newton's law shows that

$$\mathbf{F} = m\mathbf{a} \Rightarrow T\mathbf{i} + (N - mg)\mathbf{j} = -m\frac{V^2}{L}(\sin\theta\mathbf{i} + \cos\theta\mathbf{j})$$

$$\Rightarrow N = mg - m\frac{V^2}{L}\cos\theta$$

[2 POINTS]

5.3 Hence, calculate a formula for the critical speed at which your foot must leave the ground.

Note that the minimum value of N occurs for $\theta = 0$. The condition $N=0$ gives

$$V = \sqrt{gL}.$$

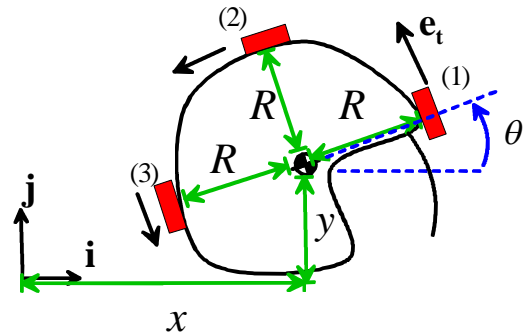
[2 POINTS]

5.4 Estimate the length of your legs, and measure your maximum walking speed experimentally. Be sure to walk so that your load-bearing leg remains straight (bent legs are illegal in speed walking – if you practice a special gait with bent legs that allows your COM to move along a straight line, you can easily exceed the theoretical limit). Explain briefly how you did the experiment. Compare your speed with the predictions of 5.3.

[4 POINTS]

8. OPTIONAL – extra credit problem Football teams are increasingly using instrumented helmets to detect head impacts that may lead to injury <http://www.browndailyherald.com/campus-news/football-helmet-sensors-measure-concussions-1.1672758>, <http://www.sportsgrid.com/nfl/nfl-looks-to-technology-to-prevent-head-injuries/>

The figure shows a helmet with three accelerometers attached to its surface. Each accelerometer measures the acceleration component in a direction parallel to its axis. It is then necessary to calculate the acceleration of the center of mass of the head a_x, a_y , together with the rotational acceleration $d^2\theta/dt^2$. This data can then be used to deduce the location, direction, and magnitude of the impact force.



7.1 Suppose that the center of mass of the head has instantaneous position $x(t), y(t)$ as shown, and consider an accelerometer that subtends an angle $\theta(t)$ to the horizontal. Write down the components of the position vector of the accelerometer.

$$\mathbf{r} = \{x(t) + R\cos\theta(t)\mathbf{i}\} + \{y(t) + R\sin\theta(t)\mathbf{j}\}$$

[1 POINT]

7.2 Write down the components of a unit vector tangent to the accelerometer. Hence, find a formula for the component of acceleration parallel to this direction ($a_t^{(1)}$), in terms of $\theta(t)$ and its time derivatives, as well as time derivatives of $x(t), y(t)$ (Mathematica will make this painless)

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In[48]:= << VectorAnalysis`

In[49]:= r = {x[t] + R * Cos[θ[t]],
              y[t] + R * Sin[θ[t]], 0};

In[50]:= v = D[r, t];

In[51]:= a = D[v, t];

In[52]:= evec = {-Sin[θ[t]], Cos[θ[t]], 0};

▼In[53]:= {-Sin[t θ], Cos[θ[t]]}
Out[53]= {-Sin[t θ], Cos[θ[t]]}

▼In[54]:= Simplify[DotProduct[a, evec]]
Out[54]= -Sin[θ[t]] x''[t] + Cos[θ[t]] y''[t] + R θ''[t]

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[3 POINTS]

7.3 Suppose that three accelerometers are mounted on the helmet, 90 degrees apart, as shown on the figure. Use the result of (7.2) to write down formulas for the tangential accelerations $a_t^{(2)}, a_t^{(3)}$ of accelerometers (2) and (3) in terms of $d^2x/dt^2, d^2y/dt^2, \theta(t), d^2\theta/dt^2$ and R .

We just need to add 90 degrees and 180 degrees to θ in the preceding problem – recall that $\cos(\theta + 90) = -\sin \theta$ and $\sin(\theta + 90) = \cos \theta$, so that

$$a_t^{(2)} = -x''(t) \cos \theta - y''(t) \sin \theta + R\theta''(t)$$

$$a_t^{(3)} = x''(t) \sin \theta - y''(t) \cos \theta + R\theta''(t)$$

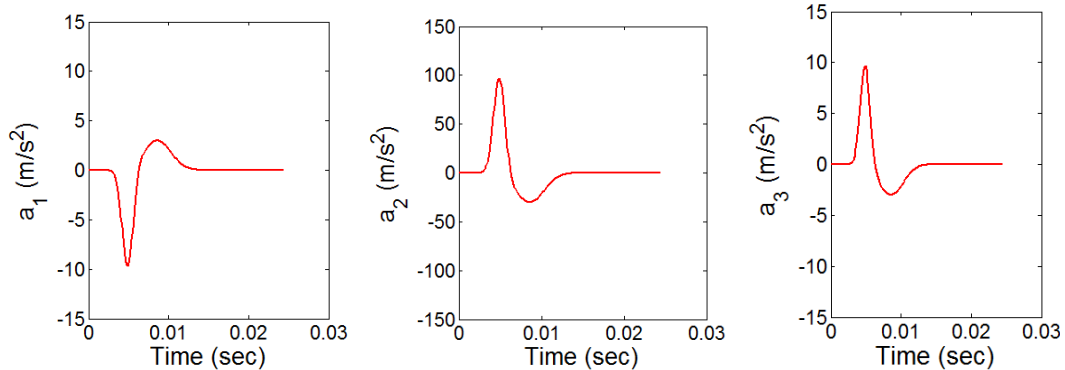
7.4 Suppose that $a_t^{(1)}, a_t^{(2)}, a_t^{(3)}$ are measured as functions of time. Derive formulas for $d^2x/dt^2, d^2y/dt^2, d^2\theta/dt^2$ in terms of these three measurements and the angle θ .

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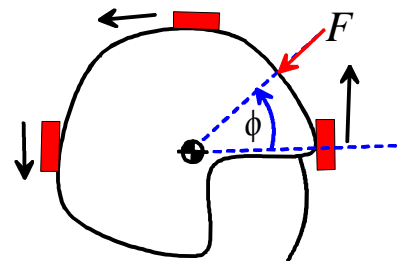
In[86]:= eq1 = a1 == -Sin[θ[t]] x''[t] + Cos[θ[t]] y''[t] + R θ''[t];
In[87]:= eq2 = a2 == -Cos[θ[t]] x''[t] - Sin[θ[t]] y''[t] + R θ''[t]
eq3 = a3 == Sin[θ[t]] x''[t] - Cos[θ[t]] y''[t] + R θ''[t]
Simplify[Solve[eq1 && eq2 && eq3,
{x''[t], y''[t], θ''[t]}]]
Out[87]= a2 == -Cos[θ[t]] x''[t] - Sin[θ[t]] y''[t] + R θ''[t]
Out[88]= a3 == Sin[θ[t]] x''[t] - Cos[θ[t]] y''[t] + R θ''[t]
Out[89]= {{x''[t] →
1/2 ((a1 - 2 a2 + a3) Cos[θ[t]] + (-a1 + a3) Sin[θ[t]]),
y''[t] → 1/2 ((a1 - a3) Cos[θ[t]] +
(a1 - 2 a2 + a3) Sin[θ[t]]), θ''[t] → (a1 + a3) / (2 R)}}

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[3 POINTS]



7.5 Suppose that a helmet starts at rest and has orientation $\theta = 0$. An impact is then detected with acceleration signals shown in the figure (note that $a_3(t) = -a_1(t)$). If the mass of the head is 4.5kg, what is the peak impact force? What is the angle ϕ of the impact location on the head?



Since $a_3(t) = -a_1(t)$ we see that $\frac{d^2\theta}{dt^2} = 0$ and since

$\theta = 0$ and the head is at rest at time $t=0$ this implies that $\theta = 0$ for all time.

This means that the expressions for the accelerations simplify to

$$x''(t) = -a_2 \quad y''(t) = a_1$$

At the max acceleration, $x''(t) = -100$ $y''(t) = -10$

The force follows as $F=ma$, i.e. $\mathbf{F} = -450\mathbf{i} - 45\mathbf{j}$.

Since the head translates without rotation during the impact the force must exert zero moment about the COM – it must act through the center of mass. This means that $\phi = \tan^{-1}(F_y / F_x) = \tan^{-1}(1/10) = 5.7^\circ$

[4 POINTS]