EN40: Dynamics and Vibrations

Homework 4: Work, Energy and Linear Momentum
Due Friday March 2nd
MAX SCORE: 76

School of Engineering
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1. The 'Morse Potential' is often used to approximate the forces acting between atoms in a diatomic molecule. The work done to separate two interacting atoms from the equilibrium distance between atoms (r_e) to a distance d apart has the following form

$$V(d) = E_0 \left[1 - \exp\{-a(d - r_e)\} \right]^2$$

1.1 Find an expression for the magnitude of the force acting between the atoms, and plot the force as a function of d, for $E_0 = 1, r_e = 1, a = 5$, with 0.9<d<3

The force follows as

$$F(d) = -\frac{dV}{dd} = 2aE_0 \left[1 - \exp\{-a(d - r_e)\} \right] \exp\{-a(d - r_e)\}$$
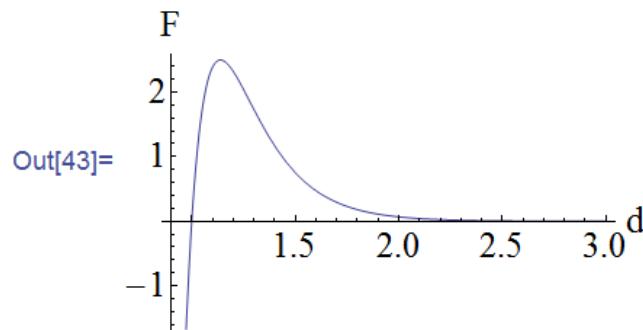
Here's the plot

In[39]:= **V = E0 * (1 - Exp[-a * (d - re)]) ^ 2**

Out[39]= $\left(1 - e^{-a\,(d-re)} \right)^2$ E0

In[40]:= **F = D[V, d] /. {a → 5, re → 1, E0 → 1}**

Out[40]= $10\, e^{-5\,(-1+d)} \left(1 - e^{-5\,(-1+d)} \right)$

In[43]:= **Plot[F, {d, 0.9, 3},**
    AxesLabel → {"d", "F"},
    AxesStyle → Directive[Black, 25]]

Out[43]=

(OK to plot graph upside down as well – that just uses a different sign convention for positive forces. In the plot shown, attractive forces acting between the atoms are taken to be positive.

[3 POINTS]

1.2 Hence, express the following physical quantities in terms of the parameters in the potential

(i) The bond strength (the force required to break the bond)

The separation corresponding to the maximum force can be computed by differentiating F with respect do d, setting the result to zero, and solving for d. Substituting this back into the expression for F then gives the maximum force. Here's the whole calculation in Mathematica

$\text{In[60]}:=$ **V = E0 * (1 - Exp[-a * (d - re)]) ^ 2**

$\text{Out[60]}= \left(1 - e^{-a\,(d-re)}\right)^2 E0$

$\text{In[61]}:=$ **F = D[V, d]**

$\text{Out[61]}= 2\,a\,e^{-a\,(d-re)}\left(1 - e^{-a\,(d-re)}\right) E0$

$\text{In[62]}:=$ **dval = d /. Solve[D[F, d] == 0, d]**

Solve::ifun :

Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

$\text{Out[62]}= \left\{ \dfrac{a\,re + \text{Log}[2]}{a} \right\}$

$\text{In[63]}:=$ **Simplify[F /. d → dval]**

$\text{Out[63]}= \left\{ \dfrac{a\,E0}{2} \right\}$

The maximum force is therefore $F_{max} = aE_0 / 2$

[2 POINTS]

(ii) The binding energy (the total energy required to pull the atoms apart from their equilibrium spacing to infinity)

The work of separation is $V(d \to \infty) - V(r_e) = E_0$.

[1 POINT]

(iii) The stiffness of the bond (i.e. the slope of the force-separation relation at the equilibrium spacing.

The stiffness is $\left. \dfrac{dF}{dd} \right|_{d=r_e} = 2a^2 E_0$

[1 POINT]

1.3 A hydrogen molecule has bond spacing 0.74A, binding energy 4.72eV (electron-volts), and stiffness 33.72 eV/A^2. Estimate the force required to break the bond. Express your answer in eV/A.

The bond stiffness and binding energy can be solved for a, giving $a = 1.89A^{-1}$. The force follows as
$$F_{max} = aE_0 / 2 = 4.46eV / A$$
Note that 1 eV/A is 1.602 10^{-9}Newtons.

[2 POINTS]

2. The figure shows the force-v-strain relation for a specimen of spider-silk (from Biophys J. 2007 December 15; 93(12): 4425–4432.). Estimate the total energy per unit length required to stretch the silk to its breaking point (use the 44% relative humidity (RH) curve).
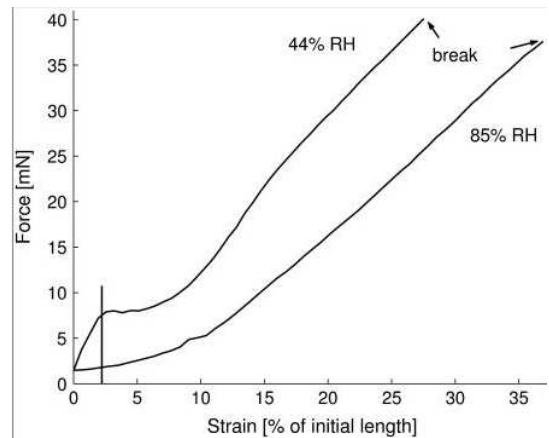
The energy required to beak a fiber is $E = \int_0^{L_f} F(x)dx$ so the energy per unit length is

$E = \int_0^{L_f} F(x)\dfrac{dx}{L_0} = \int_0^{L_f/L_0} F(\varepsilon)d\varepsilon$.   We can evaluate the integral by getting a few points off the curve. For example:

The integral can be evaluated using the 'Trapz' command in MATLAB
```
>> e = [0,2,6,7.5,10,15,26]/100;
>> F = [1.5,8,8,9,10,21.5,40]/1000;
>> trapz(e,F)
```
ans =

0.0050

ε (%)	F (mN)
0	1.5
2	8
6	8
7.5	9
10	10
15	21.5
26	40

Giving 0.005 J/m
It's OK to do a more crude estimate by just approximating the area under the curve as a triangle.

[3 POINTS]

2.2 Consider an idealized spider-web made from three radial threads, as shown in the figure. Take the length of threads to be 15cm. (A detailed study of spider-web dimensions and their correlation with the spider's diet can be found here http://deepblue.lib.umich.edu/bitstream/2027.42/31447/1/0000367.pdf). Suppose that the center of the web is impacted by (a) a house-fly, with mass 12mg and speed 4.5mph; or (b) a horse-fly with mass 12mg and speed 90mph (!! – the reference for this is

15cm
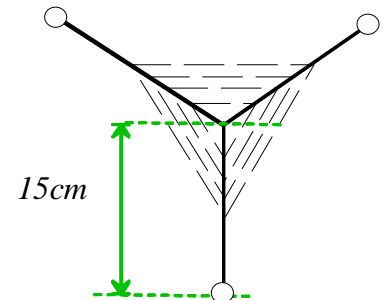
, which may not be trustworthy…).

Using energy conservation, estimate whether the flies will break the web. Assume that the flies move perpendicular to the plane of the web. Spiral threads (shown dashed in the figure) can be ignored.

> The kinetic energy of a house-fly at 4.5mph (2.011m/s) is $mv^2/2= 2.428\times10^{-5}$ J
> For a horse-fly the same calculation gives 0.0096 J
>
> The maximum energy that the web can absorb is the total length of the radial fibers multiplied by the energy per unit length at fracture, i.e. $3\times0.15\times0.005=0.00225$ J
>
> Thus the web will trap the house-fly, but the horse-fly will break the web.

[3 POINTS]

2.3 If the web remains intact, estimate the deflection of the web and the maximum force in the threads. Approximate the force-deflection curve as a straight-line for this calculation.

> We can do this calculation by making use of three ideas:
> - The web and fly together make up a conservative system. This means that the sum of the kinetic and potential energy in the system is constant.
> - If the threads have a straight-line force-extension relation, the potential energy in each of the threads after extension is $V = \dfrac{1}{2}k(L-L_0)^2$, where L is the stretched length of the thread, and L_0 is its initial length, and k is its stiffness.

Just before the fly impacts the web the total energy of the system is $T+V = mv^2/2$.
At the maximum deflection of the web, the fly comes to rest. The total energy is therefore
$T+V = \dfrac{3}{2}k(L-L_0)^2$. Since energy is conserved, the length of the threads follows as

$$L = L_0 + v\sqrt{\frac{m}{3k}}$$

The length is related to the out-of-plane deflection of the center of the web by

$$L = \sqrt{L_0^2 + z^2} \Rightarrow z = \sqrt{\left(L_0 + v\sqrt{\frac{m}{3k}}\right)^2 - L_0^2}$$

The stiffness can be estimated from the given force-strain curve. For a 15cm thread, the extension at fracture is approximately 0.15cmx27%=0.04m. The corresponding force is about 40mN, so the stiffness is about 1*N/m*.

Substituting numbers into the formula then gives z=0.035m or 3.5 cm.

The force in the web follows as $F = k(L-L_0) = kv\sqrt{\dfrac{m}{3k}} = v\sqrt{\dfrac{km}{3}} = 0.004N$

[5 POINTS]

3. Reconsider the ski-jump problem analyzed in HW 3. During flight, three forces act on the skier: lift, weight, and drag.

3.1 For each force, state whether the force is conservative, non-conservative, or workless. Explain your reasoning briefly (one or two sentences for each).

> Drag: Non-conservative. The work done by the force during motion is monotonically increasing regardless of the path, and consequently must be path dependent.
> Lift: Acts perpendicular to the direction of motion of the force. Therefore $\mathbf{F}.\mathbf{v}=0$ and the force is workless.
> Weight: Conservative – the potential energy is *mgh*.

[3 POINTS]

3.2 Write down an expression for the rate of work done by each force acting on the skier, as functions of the angle of attack α and the velocity components (or, if appropriate, the velocity magnitude).

$$P_{gravity} = -mg\mathbf{j}\cdot\left(v_x\mathbf{i}+v_y\mathbf{j}\right) = -mgv_y$$

$$P_{Lift} = F_L\mathbf{n}\cdot V\mathbf{t} = 0$$

$$P_{Drag} = -\frac{1}{2}\rho\left(-0.5072 + 0.04398\alpha - 2.861 \times 10^{-4}\alpha^2\right)V\mathbf{v}\cdot\mathbf{v}$$

$$= -\frac{1}{2}\rho\left(-0.5072 + 0.04398\alpha - 2.861 \times 10^{-4}\alpha^2\right)V^3$$

[3 POINTS]

3.3 Write down the energy conservation equation relating the skier's kinetic energy to the work done by the forces acting on the skier. Either the power or total work expression can be given…

$$P_{gravity} + P_{Drag} = \frac{dT}{dt}$$

$$W_{gravity} + W_{Drag} = T_1 - T_0$$

(T is the kinetic energy, P is the rate of work, and *W* is the total work)

[1 POINT]

3.4 Write down *two* expressions for the work done on the skier by air resistance (one in terms of a time integral, the other in terms of the kinetic and potential energy of the skier at the start and end of the jump).

> By definition, the work done by the drag force is $\Delta W = \int P_{Drag}\,dt$. This can also be expressed in the form

$$+\int P_{Drag}\,dt = \int\left(\frac{dT}{dt} - P_{gravity}\right)dt = \int\left(\frac{dT}{dt} + \frac{d(PE)}{dt}\right)dt = \Delta T + \Delta(PE)$$

[2 POINTS]

3.5 Use the MATLAB code you wrote for the ski-jump problem in HW3 (or see the solutions if you weren't able to get the code to work) to calculate the total energy dissipated by air resistance. Use both approaches in step 4, and use parameter values given in Problem 1.5 or HW3. Note that you can use the MATLAB 'Trapz' command to do the time integral. Why are the two answers different? What would

you need to do to make them closer together? You do not need to submit a full MATLAB code with your solution – just report the lines of code you added to your solution to HW3 to solve this problem.

Here's a MATLAB script to do both calculations added after the ODE solver

```
 options = odeset('Events',@event);
[times,sols] = ode45(@eom,[0,8],initial_w,options);

% This loop computes the rate of work done by drag at each time
for i =1:length(times)
    x = sols(i,1); y = sols(i,2);
    vx = sols(i,3); vy = sols(i,4);
    phi = atan(-vy/vx)*180/pi;
    alpha = psi + phi;
    Fd = 0.5*rho*(-0.5072 + 0.04398*alpha - 2.861e-4*alpha^2);
    V = sqrt(vx^2+vy^2);
    Wdrag(i) = -Fd*V^3;
End
% The trapz command integrates Wdrag with respect to time
totwork_integrated = trapz(times,Wdrag)

startke = m*(vx0^2+vy0^2)/2;
endke = m*(sols(end,3)^2+sols(end,4)^2)/2;
endpe = m*g*sols(end,2);
totwork_conservation = endke-startke + endpe
```

The integral gives the total work done as -60.143kJ
The conservation equation gives the total work done as -60.167kJ

GRADERS – note that people may get different numbers depending on the skier mass they chose – grade based on process rather than numbers.

There are two possible reasons for the discrepancy: (i) the ODE solver may not be giving a sufficiently accurate solution, and (ii) the approximation to the time integral returned by 'trapz' may not be sufficiently accurate, because there are not enough (time,work) points to describe the time variation accurately.

To improve the accuracy of the ODE solver, you can use the 'RelTol' option, as
```
options = odeset('Events',@event,'RelTol',0.00000001);
```

This changes the integrated work done to -60.165kJ but does not change the value obtained from the conservation equation. To improve the accuracy of the 'trapz' function, you can increase the number of time values in the solution vector, e.g. by
```
    [times,sols] = ode45(@eom,[0:0.001:8],initial_w,options);
```
Both methods then yield -60.167kJ

**[4 POINTS]**
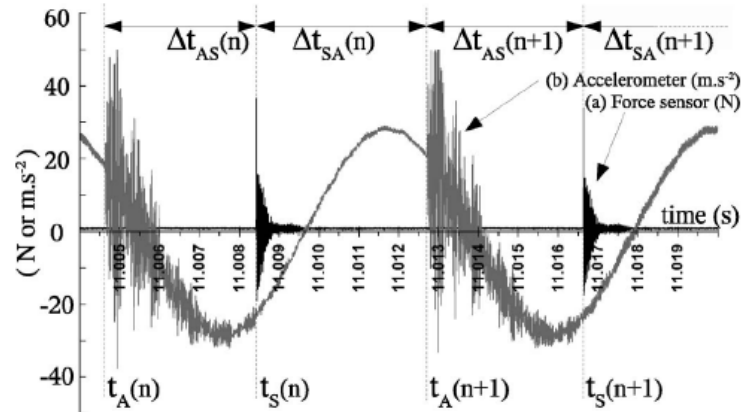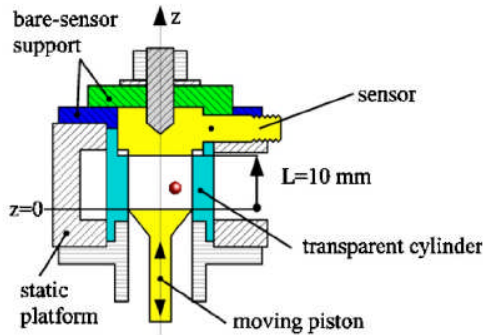
**4.** The background to this problem can be found in Leconte *et al* Applied Physics Letters, **89**, 243518 (2006). Both figures shown above are from this source. This paper describes an experimental method for measuring restitution coefficient of mm sized particles. The apparatus confines a particle of interest between two flat platens. The upper platen is stationary, while the lower vibrates vertically at a fixed frequency. An accelerometer is mounted to the vibrating platen, and a force sensor is mounted on the fixed platen. The ball bounces between the two surfaces, and the sensors detect the times at which impacts occur. The entire experiment is conducted in zero-g (using an aircraft in parabolic flight). A representative experimental result is shown in the figure above. In the test shown, the ball had a radius of 1mm, and mass $32.7 \times 10^{-6}$ kg
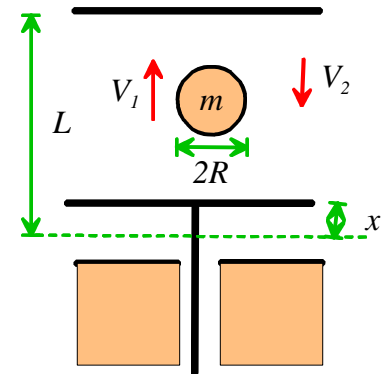
**GRADERS – Note that there is likely to be considerable variation in numbers in this problem because it is not easy to read times, etc off the graphs very accurately. Anything where the working is clearly explained and the process is correct should receive credit.**

4.1 Note that the lower platen vibrates with simple harmonic motion. Use the experimental data provided in the figure to calculate the period and frequency of the vibration, the amplitude of the velocity $V_0$ and the amplitude of the displacement $X_0$ of the platen.

Simple harmonic motion formulas give $x(t) = X_0 \sin \omega t \quad v(t) = X_0 \omega \cos \omega t \quad a(t) = -X_0 \omega^2 \sin \omega t$

The period of vibration can be read off the graph – roughly *T=0.0084*sec.
The angular frequency of vibration follows as $\omega = 2\pi / T = 748 rad / s$
The amplitude of the acceleration is about 30m/s². The amplitude of the velocity follows as 30/748=0.0401 m/s. The amplitude of the displacement follows as $0.0401/748=5.36 \times 10^{-5}$m.

**[2 POINTS]**

4.2 Use the experimental data provided to estimate the value of *x* at the time that the particle impacts the lower platen. Also, determine the velocity of the platen at the instant of impact.

The impact occurs about $t=0.003s$ after a zero-crossing of the acceleration. Note also that the acceleration is positive at the instant of impact, which means that $x$ must be negative. Therefore

$$x(t) = X_0 \sin \omega t = -5.36 \times 10^{-5} \sin(748 \times 0.003) = -4.19 \times 10^{-5} m$$

The velocity at impact follows as $v(t) = V_0 \cos \omega t = -0.0401 \cos(748 \times 0.003) = 0.025 m/s$

**[2 POINTS]**

4.3 Use the results of 4.2, together with the experimental data, to determine the velocities $V_1, V_2$ of the particle moving between the platens.

The total distance traveled by the ball is $L$-$x$-$2R$=8.0419mm
The time for the ball to travel from the bottom platen to the top is $\Delta t_{AS} = 0.004s$
The time for the ball to travel back from the top platen to the bottom is $\Delta t_{SA} = 0.0044s$

The velocities follow as $V_1$=2.011m/s  $V_2$=1.83m/s

**[2 POINTS]**

4.4 Find the restitution coefficient of the ball with both the upper and lower platens.

The upper platen is stationary.  The restitution coefficient formula gives $e = \dfrac{V_2}{V_1} = 0.91$

The lower platen is moving – its velocity may be assumed to be constant during the impact.  The restitution coefficient formula gives

$$e = \frac{V_1 - v}{V_2 + v} = 1.072$$

The restitution coefficient exceeds 1 because the platen is moving at constant speed during the impact –the impact takes place over a finite (albeit short) time, and energy is supplied by the actuator to the ball.

**[2 POINTS]**

**5.** The figure (from Kilcast "Solid Food") shows the variation of force with time when tomatoes are dropped onto a flat, stationary surface. Suppose that the variation of impact force with time can be approximated by the following function

$$F(t) = F_0 \left( \frac{t}{t_0} \right) \exp\left( -\left( \frac{t}{t_0} \right)^3 \right)$$

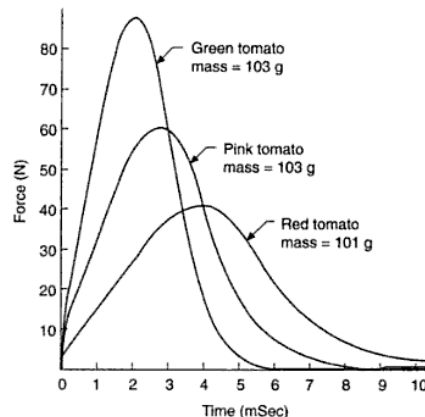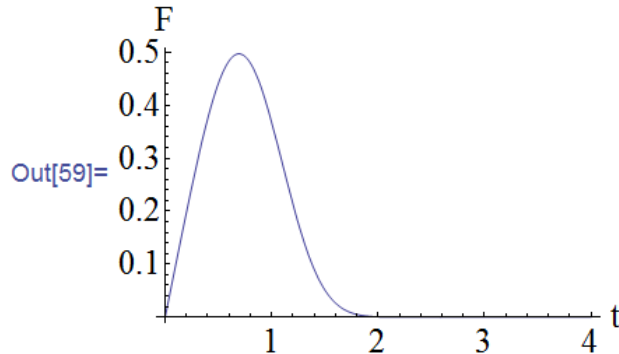where $t_0, F_0$ are two numbers that can be adjusted to give the best fit to the experimental data.



Fig. 5.12  Impact force response curves for three tomatoes of the same mass dropping 10 cm onto a rigid plate (Lichtensteiger et al., 1988).

5.1 Use Mathematica to plot $F(t)$, for $t_0 = 1, F_0 = 1$.

In[59]:= `Plot[t * Exp[-t^3], {t, 0, 4}, AxesLabel → {"t", "F"},`
`AxesStyle → Directive[Black, 25]]`

Out[59]=

**[2 POINTS]**

5.2 Find a formula for the time at which $F(t)$ is a maximum, in terms of $t_0$, and determine the corresponding maximum force, in terms of $F_0$.

We can find the max in the usual way by differentiating $F$ with respect to $t$, setting the result to zero, and solving. This can be done by hand but here's a Mathematica solution

In[60]:= `Solve[D[t / t0 * Exp[-(t / t0)^3], t] == 0, t]`

Out[60]= $\left\{ \left\{ t \to -\left(-\frac{1}{3}\right)^{1/3} t0 \right\}, \left\{ t \to \frac{t0}{3^{1/3}} \right\}, \left\{ t \to \frac{(-1)^{2/3} t0}{3^{1/3}} \right\} \right\}$

Only the second root given here is physically relevant. Here's how to substitute this value into the expression for force in Mathematica

In[88]:= `Solve[D[t / t0 * Exp[-(t / t0)^3], t] == 0, t]`

Out[88]= $\left\{ \left\{ t \to -\left(-\frac{1}{3}\right)^{1/3} t0 \right\}, \left\{ t \to \frac{t0}{3^{1/3}} \right\}, \left\{ t \to \frac{(-1)^{2/3} t0}{3^{1/3}} \right\} \right\}$

In[89]:= `F0 * (t / t0) * Exp[-(t / t0)^3] /. %[[2]]`

Out[89]= $\dfrac{F0}{(3\,e)^{1/3}}$

**[2 POINTS – Mathematica not required, of course]**

5.3 Find a formula for the impulse exerted by the force, in terms of $t_0, F_0$ (use Mathematica to do the integral – you can use Assuming[assumption,expression] to remove the annoying conditional in the result. Check the Mathematica help for more information the Assuming[] command).

By definition, the impulse is $I = \int_0^T F(t)dt$. In this case we can take the upper limit $T$ to be infinite.

The integral can be done in Mathematica.

```
In[90]:= Assuming[t0 > 0, Integrate[F0 * (t / t0) * Exp[-(t / t0)^3],
           {t, 0, Infinity}]]

Out[90]= 1/3 F0 t0 Gamma[2/3]

In[91]:= N[%]

Out[91]= 0.451373 F0 t0
```

Here Gamma is a special function – it's a sort of generalized factorial (specifically $\Gamma(n) = (n-1)!$ – it is defined as $\Gamma(z) = \int_0^\infty x^{(z-1)} e^{-x} dx$. For integer z you can evaluate the integral by successive integrations by parts, which will show the connection to a factorial)

[3 POINTS]

5.4 Determine values for t_0, F_0 that will approximate the three sets of experimental data.

We can read off the peak and the time at which the peak occurs for each curve, and use the result to estimate t_0, F_0. Roughly

- Green tomatoes:
  $F_{max} \approx 85N$, $t_{max} \approx 2ms \Rightarrow t_0 = t_{max} \times 3^{1/3} = 2.88ms$ $F_0 = F_{max}(3\exp(1))^{1/3} = 171.1N$

- Pink tomatoes
  $F_{max} \approx 60N$, $t_{max} \approx 2.8ms \Rightarrow t_0 = t_{max} \times 3^{1/3} = 4.03ms$ $F_0 = F_{max}(3\exp(1))^{1/3} = 120.8N$
- Red tomatoes
  $F_{max} \approx 40N$, $t_{max} \approx 4.2ms \Rightarrow t_0 = t_{max} \times 3^{1/3} = 6.06ms$ $F_0 = F_{max}(3\exp(1))^{1/3} = 80.51N$

[3 POINTS]

5.5 Hence, estimate the value of the restitution coefficient for red, pink and green tomatoes.

The tomatoes are dropped onto a flat horizontal surface from a 10cm height. We can use energy conservation to calculate the velocity just before impact

$$mgh = \frac{1}{2}mv_0^2 \Rightarrow v_0 = \sqrt{2gh} = 1.4m/s \text{ (downwards)}$$

We can use the impulse-momentum relation to calculate the velocity of the fruit just after impact. Take **j** to be positive upwards, then

$$\int_0^T F(t)dt\mathbf{j} = m(v_1\mathbf{j} - (-v_0)\mathbf{j}) \Rightarrow v_1 = \frac{1}{m}\int_0^T F(t)dt - v_0 = 0.45137\frac{F_0 t_0}{m} - v_0$$

The restitution coefficient follows as $e = -v_1 / (-v_0) = v_1 / v_0$

The three cases give

- Green tomatoes ($m=103$ gram) 0.759 m/s $e = 0.5421$
- Pink ($m=103$ gram) 0.7338m/s $e = 0.5235$
- Red ($m=101$ gram) : 0.780m/s $e = 0.557$

There are likely to be some variations in the predictions in student solutions, depending on what numbers are used for t_0, F_0

[3 POINTS]

6. The figure shows an apparatus to measure the impulse exerted by a sub-surface explosive device. It consists of a piston with mass m supported by a frame. The system is initially at rest. The explosion then propels the piston vertically, and its maximum height h is measured. Derive an expression that relates the piston mass m and the height h to the impulse I exerted on the piston by the explosion. Friction can be neglected.
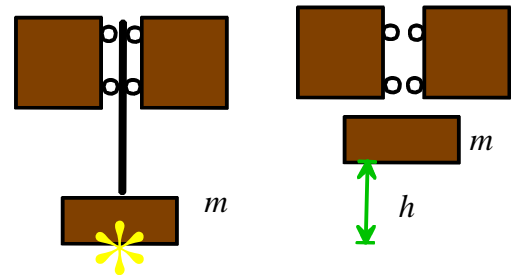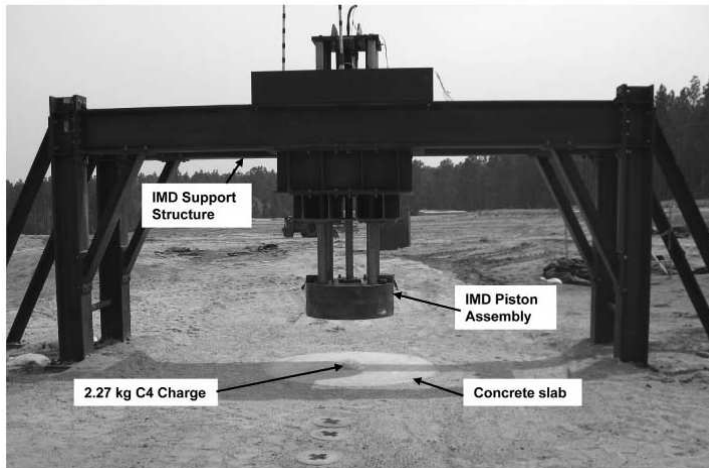
(Figure from Ehrgott, et al *Experimental Techniques,* doi: 10.1111/j.1747-1567.2009.00604.x)

Just after the explosion, the mass is traveling vertically with speed V. During subsequent flight, energy is conserved, so $mV^2 / 2 = mgh \Rightarrow V = \sqrt{2gh}$.

The impulse-momentum equation then gives $\mathbf{I} = \mathbf{p}_1 - \mathbf{p}_0 \Rightarrow I\mathbf{j} = mV\mathbf{j} \Rightarrow I = m\sqrt{2gh}$.

[5 POINTS]

7. The figure shows two masses which roll freely on the surfaces of a wedge. The two masses are tethered by a massless cable that passes over a frictionless pulley. The wedge rolls freely over a horizontal surface. At time $t=0$ the system is at rest and mass A is a distance d from the top of the wedge. The goal of this problem is to calculate the time taken for the mass at A to reach the top of the ramp. (Why anyone would care about this is beyond me, but I spent many years solving problems like this as a student… Now I have a wonderful family, many cats, and a substantial job, so you might like to see if it works for you…)
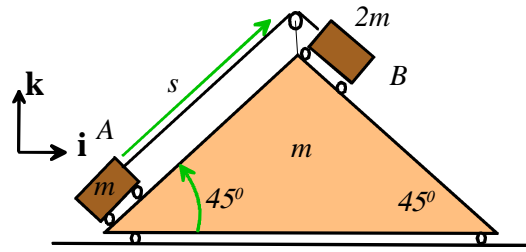
Graders – it's very fiddly getting the numbers correct on this problem – it needs very organized and systematic calculations (or Mathematica). Deduct a point or two if the final answer is not correct, but don't worry too much about the numbers. Solutions that are procedurally correct should get mostly full credit. Anyone submitting kittens, spouses, or children (or pictures thereof) should get an extra credit point.

7.1 Suppose that the wedge has horizontal speed v. Write down the velocity vectors of each small mass in terms of v and the time derivative of the distance s of the lighter of the two masses from the top of the wedge

If you have trouble writing down the velocities by inspection, try writing down the position vector of each mass and differentiating the position vector with respect to time. For example, if the tip of the wedge has position $x\mathbf{i}+h\mathbf{j}$, the position of mass A is $\mathbf{r} = x\mathbf{i}-s\cos(45)\mathbf{i}+(h-s\sin(45))\mathbf{j}$

$$\mathbf{v}_A = v\mathbf{i} - \frac{ds}{dt}\cos 45\mathbf{i} - \frac{ds}{dt}\sin 45\mathbf{j}$$

$$\mathbf{v}_B = v\mathbf{i} - \frac{ds}{dt}\cos 45\mathbf{i} + \frac{ds}{dt}\sin 45\mathbf{j}$$

[2 POINTS]

7.2 Hence, write down the total momentum, and energy of the system

The momentum is

$$\mathbf{p} = mv\mathbf{i} + m\left(v\mathbf{i} - \frac{ds}{dt}\cos 45\mathbf{i} - \frac{ds}{dt}\sin 45\mathbf{j} \right) + 2m\left(v\mathbf{i} - \frac{ds}{dt}\cos 45\mathbf{i} + \frac{ds}{dt}\sin 45\mathbf{j} \right)$$

The total energy (kinetic+potential) is

$$T + V = -mgs\sin 45 - 2mg(d-s)\sin 45$$

$$+ \frac{1}{2}mv^2 + \frac{1}{2}m\left(v - \frac{ds}{dt}\cos 45 \right)^2 + \frac{1}{2}m\left(\frac{ds}{dt}\sin 45 \right)^2 + m\left(v - \frac{ds}{dt}\cos 45 \right)^2 + m\left(\frac{ds}{dt}\sin 45 \right)^2$$

Other results are possible depending on the datum used to measure the heights of the various masses. It doesn't matter what is chosen (and you can even use a different datum for each mass), as long as everything is consistent.

[2 POINTS]

7.3 Use momentum and energy conservation to obtain a differential equation for s. Your equation should have the form

$$\frac{d^2s}{dt^2} = nasty \ \ formula$$

The system is at rest at time $t=0$ and therefore its momentum is zero. Since no external forces act on the system in the **i** direction, momentum must be conserved in the **i** direction. This means that

$$0 = mv + m\left(v - \frac{ds}{dt}\cos 45\right) + 2m\left(v - \frac{ds}{dt}\cos 45\right)$$

We can solve this equation for v, with the result

$$v = \frac{3}{4\sqrt{2}}\frac{ds}{dt}$$

The initial total energy is $-mgd \sin(45)$, and remains constant. Therefore

$$-mgd \sin 45 = -mgs \sin 45 - 2mg(d-s)\sin 45$$

$$+ \frac{1}{2}mv^2 + \frac{1}{2}m\left(v - \frac{ds}{dt}\cos 45\right)^2 + \frac{1}{2}m\left(\frac{ds}{dt}\sin 45\right)^2 + m\left(v - \frac{ds}{dt}\cos 45\right)^2 + m\left(\frac{ds}{dt}\sin 45\right)^2$$

Substituting for v in this equation and simplifying gives

$$0 = -mg(d-s)\sin(45) + \frac{15}{16}m\left(\frac{ds}{dt}\right)^2$$

$$\left(\frac{ds}{dt}\right)^2 = \frac{16g}{15\sqrt{2}}(d-s)$$

(anything algebraically equivalent should get credit, of course, and other solutions with a different datum for the potential energy may have a constant added to the RHS)

To get the differential equation in the form requested, differentiate both sides with respect to time

$$2\left(\frac{ds}{dt}\right)\frac{d^2s}{dt^2} = -\frac{16g}{15\sqrt{2}}\frac{ds}{dt} \Rightarrow \frac{d^2s}{dt^2} = -\frac{8}{15\sqrt{2}}g$$

[4 POINTS]

7.4 Solve the result of 7.3 to calculate the time. Then go off and start a family, get some kittens, etc.

You could do this two ways – the differential equation from the previous part just says that $d^2s/dt^2 = const$ - this is therefore just a constant acceleration problem. We can write down the formula for s as a function of time by noting that at time $t=0$ the system is at rest (no velocity) and at time $t=0$ $s=d$. Therefore

$$s = d - \frac{1}{2}\frac{8}{15\sqrt{2}}t^2$$

From the constant acceleration formula. (x=x0+v0t-a t^2/2). The time to reach $s=0$ follows as

$$t = \sqrt{\frac{15\sqrt{2}}{4g}d} = \sqrt{\frac{15d}{2\sqrt{2}g}}$$

If you prefer, you can also use the preceding formula for ds/dt, separate variables and integrate, noting that $s=d$ at $t=0$, and $s=0$ when the mass reaches the top of the ramp

$$\left(\frac{ds}{dt}\right)^2 = \frac{16g}{15\sqrt{2}}(d-s) \Rightarrow \frac{ds}{dt} = \sqrt{\frac{16g}{15\sqrt{2}}(d-s)}$$

$$\int_d^0 \frac{ds}{\sqrt{d-s}} = \sqrt{\frac{16g}{15\sqrt{2}}}\int_0^t dt \Rightarrow 2\sqrt{d} = \sqrt{\frac{16g}{15\sqrt{2}}}t \Rightarrow t = \sqrt{\frac{15\sqrt{2}}{4g}}d$$

The whole calculation can also be done in Mathematica (but some of the syntax involved in extracting solutions into a useable form is painful)

```
In[34]:= eq1 = m * v + m * (v - s' / Sqrt[2]) + 2 * m * (v - s' / Sqrt[2]) == 0;

In[35]:= vsol = v /. Solve[eq1, v];

In[32]:= eq2 =
    - m * g * d / Sqrt[2] == - m * g * s / Sqrt[2] - 2 * m * g * (d - s) / Sqrt[2] + m * v^2 / 2 +
        m * (v - s' / Sqrt[2])^2 / 2 + m * (s' / Sqrt[2])^2 / 2 + m * (v - s' / Sqrt[2])^2 +
        m * (s' / Sqrt[2])^2 /. v → vsol;

In[36]:= Simplify[Solve[eq2, s']]
```

$$\text{Out[36]= } \left\{\left\{s' \to -\frac{2 \times 2^{3/4}\sqrt{g\ (d-s)}}{\sqrt{15}}\right\}, \left\{s' \to \frac{2 \times 2^{3/4}\sqrt{g\ (d-s)}}{\sqrt{15}}\right\}\right\}$$

```
In[37]:= ode = {s'[t] == 2 * 2^(3/4) * Sqrt[g * (d - s[t]) / 15]}
```

$$\text{Out[37]= } \left\{s'[t] == \frac{2 \times 2^{3/4}\sqrt{g\ (d - s[t])}}{\sqrt{15}}\right\}$$

```
In[38]:= IC = {s'[0] == 0};

In[39]:= DSolve[Join[ode, IC], {s}, t];

In[40]:= sfunc = s /. %[[1]]
```

$$\text{Out[40]= } \text{Function}\left[\{t\}, \frac{1}{15}\left(15\,d - 2\sqrt{2}\ g\,t^2\right)\right]$$

```
In[41]:= Solve[sfunc[t] == 0, t]
```

$$\text{Out[41]= } \left\{\left\{t \to -\frac{\sqrt{15}\ \sqrt{d}}{2^{3/4}\sqrt{g}}\right\}, \left\{t \to \frac{\sqrt{15}\ \sqrt{d}}{2^{3/4}\sqrt{g}}\right\}\right\}$$

[2 POINTS]
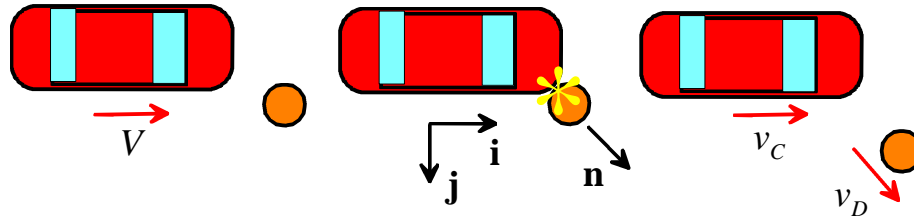
8. The figure shows a plan view of a vehicle with mass M and initial speed V colliding with one of the sand-filled drums in a 'Fitch Barrier' http://www.google.com/patents?hl=en&lr=&vid=USPAT3880404&id=3YoyAAAAEBAJ&oi=fnd&dq=fitch+barrier&printsec=abstract#v=onepage&q&f=false. The drum has mass m.

The car rolls freely, so the combined momentum of the car and drum is conserved in the $\mathbf{i}$ direction (parallel to the vehicle's motion) during impact. Friction forces between the car's wheels and the road ensure that the car continues to move in the $\mathbf{i}$ direction after impact. Friction between the drum and road during the impact may be neglected. After the impact, the car has velocity vector $v_c \mathbf{i}$, while the drum has velocity vector $v_D \mathbf{n}$, where $\mathbf{n}$ is the normal to the plane of contact.

8.1 Assume that the impact occurs with the normal vector to the contact plane at 45 degrees, and is frictionless. Write down the unit vector $\mathbf{n}$ as components in the $\{\mathbf{i},\mathbf{j}\}$ basis. Hence, determine the components of velocity in a direction parallel to $\mathbf{n}$ of the car and drum before and after impact, in terms of V, v_D and v_c

This is simple vector stuff $\mathbf{n} = (\mathbf{i}+\mathbf{j})/\sqrt{2}$

The velocity components along $\mathbf{n}$ are $\quad V\mathbf{i}\cdot\mathbf{n} = V/\sqrt{2} \qquad v_c\mathbf{i}\cdot\mathbf{n} = v_c/\sqrt{2} \qquad v_D\mathbf{n}\cdot\mathbf{n} = v_D$

[2 POINTS]

8.2 Assume that the collision has restitution coefficient e. Write down the restitution coefficient formula relating V, v_D and v_c

The normal components of velocity are related by the usual restitution coefficient formula – so
$$\frac{v_D - v_C/\sqrt{2}}{V/\sqrt{2}} = e$$

[1 POINT]

8.3 Use momentum conservation and 8.2 to determine a formula for the speed of the car v_c after collision, in terms of V, e, and the masses of the car and drum.

Momentum is conserved in the $\mathbf{i}$ direction, which requires
$$MV = Mv_c + mv_D\mathbf{n}\cdot\mathbf{i} = Mv_c + mv_D/\sqrt{2}$$

From 8.2, we see that $v_D = v_C / \sqrt{2} + eV / \sqrt{2}$, and substituting this into the preceding formula gives

$$MV = Mv_c + m(v_c + eV)/2 \Rightarrow v_c = (M - em/2)V / (M + m/2)$$

[2 POINTS]

8.4 Suppose that the car hits a series of drums with mass $m=M/5$ and $e=1/2$. How many drums are required to reduce the car's speed by a factor of 10?

$$v_c = \frac{1-1/10}{1+1/10}V = \frac{9}{11}V$$

The number of drums to reduce the car's speed by a factor of 10 satisfies the equation

$$V\left(\frac{9}{11}\right)^n = \frac{V}{10} \Rightarrow n = \log(1/10)/\log(9/11) = 12$$

[2 POINTS]