ENGN0040: Dynamics and Vibrations

Homework 5: Free and damped vibrations

School of Engineering
Brown University

Warm-up problems:

1. Complex numbers

1.1 Show that

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos(n\theta) + \mathrm{i}\,\sin(n\theta).$$

1.2 Use Euler's formula to show that

$$\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos 3\theta$$

1.3 Calculate all the values of i^{i}.

2. An automobile having a mass of 2000 kg deflects the suspension springs 0.02 m under static conditions. Determine the natural frequency of the automobile in the vertical direction, assuming damping to be negligible.

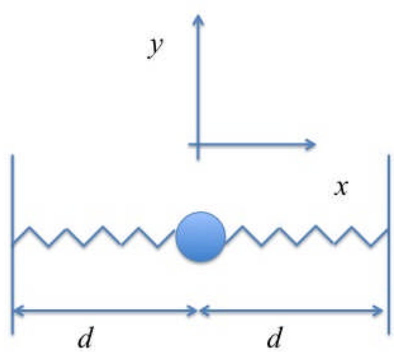
3. The radio station AM 1290 WRNI in Providence transmits at a carrier frequency f_c=1290 kHz. The amplitude of this signal is modulated with a sinusoidal oscillation f_m=1200 Hz:

$$E(t) = E_0\left[1 + \frac{1}{2}\cos(2\pi f_m t)\right]\cos(2\pi f_c t).$$

3.1 Plot E(t) as a function of time.
3.2 Show that E(t) is equivalent to the superposition of three constant-amplitude signals.
3.3 Using the Play command in Mathematica (and a computer with a speaker!), determine the audible range of frequencies for humans. What bandwidth (range of frequencies) is required to transmit the complete audible range?

More involved problems:

4. An object of mass M rests on a frictionless horizontal surface. Two identical springs of spring constants k and relaxed length l_0 are attached to the mass as shown below. The object is at rest in static equilibrium when each spring is of length d ($d>l_0$).



4.1 The mass is given a displacement of x_0 to the right. Give equations for F_1 and F_2, the forces exerted by the two springs. Use the sign convention that a positive force acts toward $+x$.

4.2 The mass is released from its position at $x=x_0$. The initial velocity is zero. Write the differential equation of motion for the mass moving in the x direction.

4.3 Solve the equation of motion, using the initial conditions.

4.4 Now suppose the mass again brought to its equilibrium position, and given a small displacement y_0 in the y direction. The displacement is so small that the lengths of the springs may be considered to be d. What is the net force acting on the mass? Give both magnitude and direction.

4.5 What is the natural frequency of the small oscillations along the y axis?

4.6 Now redo 4.2 using energy methods. Assume the mass only moves along the x direction.

5. In this problem we will get some experience with using complex numbers to solve the damped free oscillator problem. Writing $z=x+iy$, suppose z satisfies the damped oscillator equation in standard form,

$$\ddot{z}+2\varsigma\omega_{\mathrm{n}}\dot{z}+\omega_{\mathrm{n}}^2 z=0.$$

Assume $0\le\varsigma<1$.

5.1 Set $z=A\exp(ipt)$, where A is a complex constant (with magnitude and phase), and solve the resulting quadratic equation for p.

5.2 Take the real part of z to find

$$x=B\exp(-\varsigma\omega_{\mathrm{n}}t)\cos(\omega_{\mathrm{d}}t+\phi),$$

where B and ϕ are undetermined (real) constants, and $\omega_{\mathrm{d}}=\omega_{\mathrm{n}}\sqrt{1-\varsigma^2}$.

5.3 Supposing $x(0)=x_0$ and $\dot{x}(0)=0$, show that

$$x=\frac{x_0}{\sqrt{1-\varsigma^2}}\exp(-\varsigma\omega_{\mathrm{n}}t)\cos(\omega_{\mathrm{d}}t+\phi),$$

with $\tan\phi=-\varsigma/\sqrt{1-\varsigma^2}$.

5.4 The logarithmic decrement

$$\delta=\log\left[x(t_k)/x(t_{k+1})\right],$$

where the t_k are the times of the local maxima, is a measure of the damping. Show that

$$\varsigma=\frac{\delta}{\sqrt{4\pi^2+\delta^2}}.$$

6. Using MATLAB, calculate how the period of an undamped pendulum depends on the initial amplitude θ_0 of the angle of the pendulum. Assume the pendulum starts from rest. Plot your answer for the period using appropriate dimensionless variables.

7. The figure below shows a pendulum in which the support point, with mass m_1, is free to slide back and forth without friction along a horizontal bar. The mass at the end of the pendulum is m_2. There is no initial velocity in the system, and there is no damping. The goal of this problem is to find the natural frequency.

7.1 Write down the total energy as a function of x and θ, and/or their time derivatives.
7.2 What principle relates $\dot{x}$ to θ and $\dot{\theta}$? Use this principle to find $\dot{x}$ in terms of θ and $\dot{\theta}$.
7.3 Write the total energy as a function of θ and $\dot{\theta}$.
7.4 Expand the energy for small angle. You need to go to second order in the small quantities.
7.5 Using the fact that the energy is constant for this conservative system, derive the equation of motion for small oscillations. What is the natural frequency?
7.6 Consider your answer in two different limits, $m_1 >> m_2$, and $m_1 << m_2$. Can you explain why the natural frequency takes the value it does in these two limits?