School of Engineering
Brown University

ENGN0040: Dynamics and Vibrations

Homework 5: Free and damped vibrations---solutions

Warm-up problems:

1. Complex numbers

1.1 Show that

$$\left(\cos\theta + \mathrm{i}\sin\theta\right)^n = \cos\left(n\theta\right) + \mathrm{i}\,\sin\left(n\theta\right).$$

solution [1 point]:

$$\left(\mathrm{e}^{\mathrm{i}\theta}\right)^n = \mathrm{e}^{\mathrm{i}\theta n} = \cos(n\theta) + \mathrm{i}\sin(n\theta)$$

1.2 Use Euler's formula to show that

$$\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos 3\theta$$

solution [1 point]:

$$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$$

implies that

$$\cos\theta = (\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta})/2.$$

Thus,

$$\cos^3\theta = \left(\frac{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}}{2}\right)^3 = \frac{\mathrm{e}^{3\mathrm{i}\theta}}{8} + \frac{3\mathrm{e}^{\mathrm{i}\theta}}{8} + \frac{3\mathrm{e}^{-\mathrm{i}\theta}}{8} + \frac{\mathrm{e}^{-3\mathrm{i}\theta}}{8} = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta.$$

1.3 Calculate all the values of i^{i}.

solution [1 point]:

Using, Euler's formula, we can write i as a phase: $\mathrm{i} = \mathrm{e}^{\mathrm{i}\pi/2}$.

Note that we can multiply i by one by multiplying by any integer power of

$$\mathrm{e}^{\mathrm{i}2\pi k} = 1,$$

where k is an integer. Thus,

$$\mathrm{i}^{\mathrm{i}} = \mathrm{e}^{-(\pi/2 + 2\pi k)}.$$

Note that there are infinitely many values, one corresponding to each k. All the values are real.

2. An automobile having a mass of 2000 kg deflects the suspension springs 0.02 m under static conditions. Determine the natural frequency of the automobile in the vertical direction, assuming damping to be negligible.

solution [1 point]:

In static equilibrium, $k\delta = mg$, where k is the effective spring constant of the springs, mg is the car weight, and δ is the static deflection. Thus

$$\omega_n = \sqrt{g/\delta} \approx \sqrt{9.81/0.02} \quad \text{rad/s} \approx 22.15 \text{ rad/s} \approx 3.525 \quad \text{Hz}$$

3. The radio station AM 1290 WRNI in Providence transmits at a carrier frequency f_c=1290 kHz. The amplitude of this signal is modulated with a sinusoidal oscillation f_m=1200 Hz:

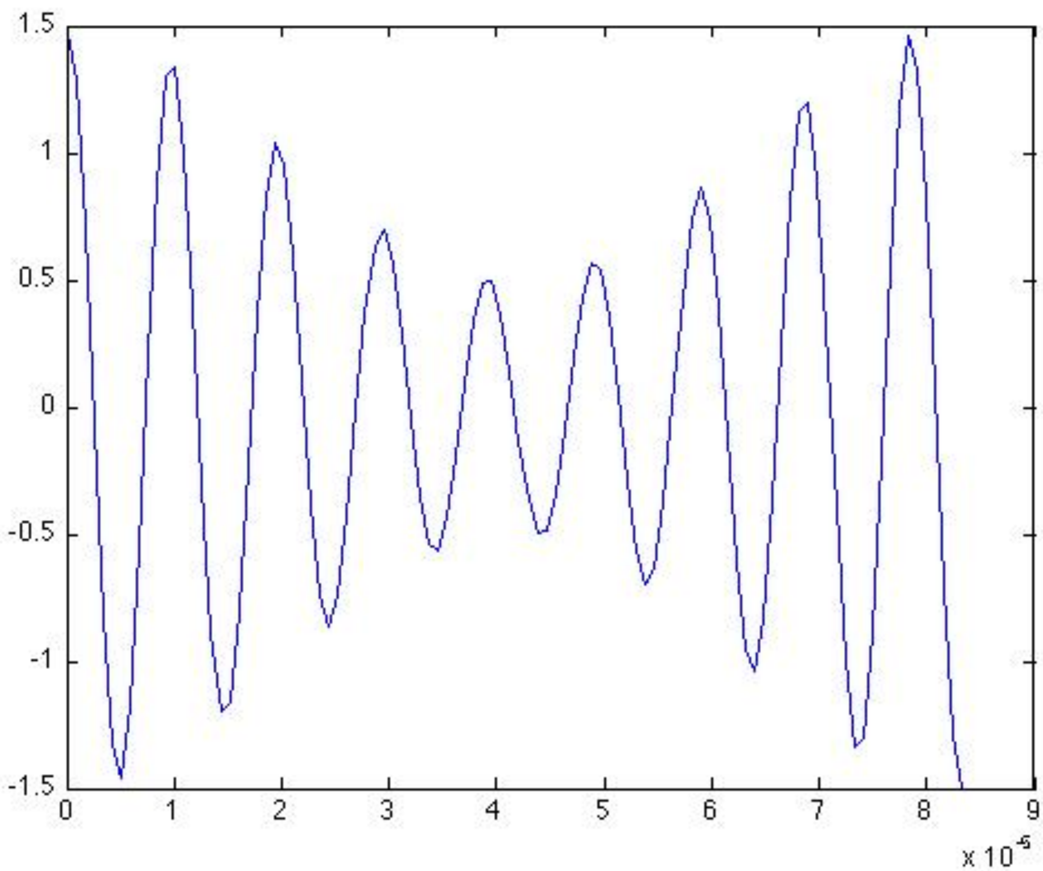
$$E(t) = E_0\left[1+\frac{1}{2}\cos\left(2\pi f_m t\right)\right]\cos\left(2\pi f_c t\right).$$

3.1 Plot E(t) as a function of time.

solution [1 point]

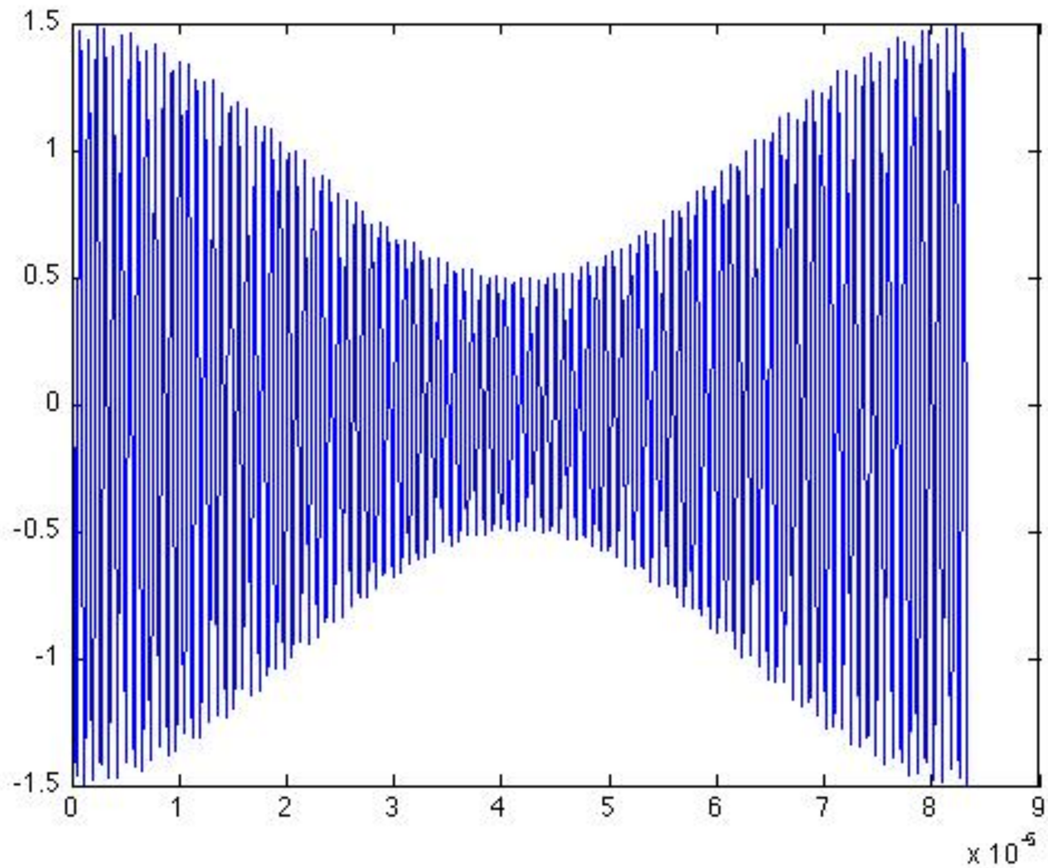
There is about a 1000-fold difference in the carrier frequency and the modulation frequency. The carrier frequency oscillates extremely rapidly compared to the modulating signal. If we try to plot the signal without thinking carefully about how fast the carrier signal oscillates, we will get a misleading plot. Here is some MATLAB code and the output for the E(t)/E_0 vs time for one period of the modulating signal:

```
fc=1290000;fm=12000;
t=linspace(0,1/fm,100);
y=(1+1/2*cos(2*pi*fm*t)).*cos(2*pi*fc*t);
plot(t,y)
```



The problem here is that `linspace` by default sets up a vector of 100 points. The graph above is misleading because the carrier signal has a period that is comparable to the spacing between the points in the vector `t`. The solution is to add more points:

```
fc=1290000;fm=12000;
t=linspace(0,1/fm,1000);
y=(1+1/2*cos(2*pi*fm*t)).*cos(2*pi*fc*t);
plot(t,y)
```



We could add even more points to make this more accurate, but at least this figure realistically conveys the discrepancy in frequencies.

3.2 Show that E(t) is equivalent to the superposition of three constant-amplitude signals.

solution [1 point]

$\cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos\alpha\cos\beta.$

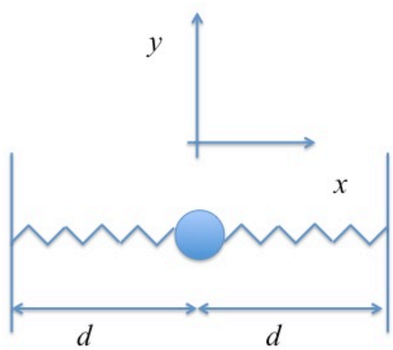
Thus,

$$E(t)/E_0=\cos\left(2\pi f_c t\right)+\cos\left[2\pi\left(f_c+f_m\right)t\right]+\cos\left[2\pi\left(f_c-f_m\right)t\right]$$

3.3 Using the Play command in Mathematica (and a computer with a speaker!), determine the audible range of frequencies for humans. What bandwidth (range of frequencies) is required to transmit the complete audible range?
solution [1 point]
The human auditory range is from about 20 Hz to 20 kHz. Thus, the bandwidth is about 40 kHz, or from about 1250 kHz to 1330 kHz.

More involved problems:

4. An object of mass M rests on a frictionless horizontal surface. Two identical springs of spring constants k and relaxed length l_0 are attached to the mass as shown below. The object is at rest in static equilibrium when each spring is of length d ($d>l_0$).



4.1 The mass is given a displacement of x_0 to the right. Give equations for F_1 and F_2, the forces exerted by the two springs. Use the sign convention that a positive force acts toward $+x$.
solution [1 point] Label the spring on the left by 1 and the spring on the right by 2.

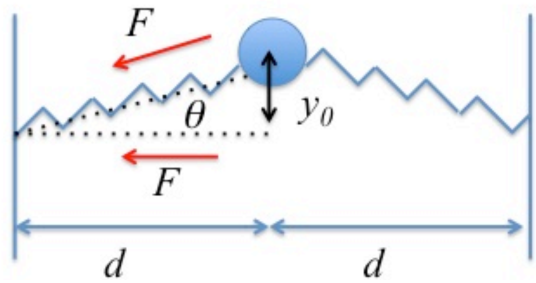
$$F_1 = -k\left(d + x_0 - l_0\right),$$
$$F_2 = k\left(d - x_0 - l_0\right).$$

4.2 The mass is released from its position at $x=x_0$. The initial velocity is zero. Write the differential equation of motion for the mass moving in the x direction.
solution [1 point]
$m\ddot{x} = -2kx.$

4.3 Solve the equation of motion, using the initial conditions.
solution [1 point]

$$x = x_0 \cos\left(\sqrt{\frac{2k}{m}}t\right).$$

4.4 Now suppose the mass again brought to its equilibrium position, and given a small displacement y_0 in the y direction. The displacement is so small that the lengths of the springs may be considered to be d. What is the net force acting on the mass? Give both magnitude and direction.
solution [1 point]
The springs now make a small angle θ with the x-axis. Thus, within an accuracy of order θ, the length of each spring does not change. Thus, the tension in each spring acts parallel to the spring, directed away from the central mass, and with the same magnitude $F=k(d-l_0)$ (to accuracy of order θ) as when $\theta=0$. Since $\theta \sim y_0/d$, the net force is $F_{net}=-2k(d-l_0)y_0/d$.



4.5 What is the natural frequency of the small oscillations along the y axis?
solution [1 point]

$$\omega_\mathrm{n} = \sqrt{\frac{2k}{m}\frac{d-l_0}{d}}.$$

4.6 Now redo 4.2 using energy methods. Assume the mass only moves along the x direction.
solution [1 point]

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k\left(d+x-l_0\right)^2 + \frac{1}{2}k\left(d-x-l_0\right)^2$$

$$\dot{E} = \left[m\ddot{x} + k\left(d+x-l_0\right) - k\left(d-x-l_0\right)\right]\dot{x}$$

But $\dot{E} = 0$ so

$$m\ddot{x} + 2kx = 0$$

5. In this problem we will get some experience with using complex numbers to solve the damped free oscillator problem. Writing $z=x+\mathrm{i}y$, suppose z satisfies the damped oscillator equation in standard form,

$$\ddot{z} + 2\varsigma\omega_\mathrm{n}\dot{z} + \omega_\mathrm{n}^2 z = 0.$$

Assume $0 \le \varsigma < 1$.

5.1 Set $z = A\exp(\mathrm{i}pt)$, where A is a complex constant (with magnitude and phase), and solve the resulting quadratic equation for p.
solution [1 point]
The differential equation becomes

$$\left(-p^2 + 2\mathrm{i}\varsigma\omega_\mathrm{n}p + \omega_\mathrm{n}^2\right)z = 0.$$

The quantity in parentheses must vanish. Therefore,

$$p = \frac{-2\mathrm{i}\varsigma\omega_\mathrm{n} \pm \sqrt{4\omega_\mathrm{n}^2 - 4\varsigma^2\omega^2}}{-2}$$

$$p = \mathrm{i}\varsigma\omega_\mathrm{n} \pm \omega_\mathrm{n}\sqrt{1-\varsigma^2}$$

$$p = \mathrm{i}\varsigma\omega_\mathrm{n} \pm \omega_\mathrm{d}$$

5.2 Take the real part of z to find

$$x = B\exp(-\varsigma\omega_\mathrm{n}t)\cos(\omega_\mathrm{d}t + \phi),$$

where B and ϕ are undetermined (real) constants, and $\omega_{\mathrm{d}} = \omega_{\mathrm{n}}\sqrt{1-\varsigma^2}$.

solution [1 point]

$z = A\mathrm{e}^{\mathrm{i}pt} = A\exp\left(-\varsigma\omega_n t\right)\exp\left(\pm\mathrm{i}\omega_d t\right).$

Since A is complex, we may write $A = B\exp\left(\mathrm{i}\phi\right)$, where B and ϕ are real. Thus,

$x = B\exp(-\varsigma\omega_{\mathrm{n}} t)\cos(\omega_{\mathrm{d}} t + \phi).$

5.3 Supposing $x(0) = x_0$ and $\dot{x}(0) = 0$, show that

$$x = \frac{x_0}{\sqrt{1-\varsigma^2}}\exp(-\varsigma\omega_{\mathrm{n}} t)\cos(\omega_{\mathrm{d}} t + \phi),$$

with $\tan\phi = -\varsigma / \sqrt{1-\varsigma^2}$.

solution [1 point]

$x(0) = x_0 \Rightarrow x_0 = B\cos\phi.$

Also,

$\dot{x}(0) = 0 \Rightarrow -\varsigma\omega_{\mathrm{n}}\cos\phi - \omega_{\mathrm{d}}\sin\phi = 0.$

Thus,

$\tan\phi = -\frac{\varsigma}{\sqrt{1-\varsigma^2}}.$

You can draw a right triangle with angle ϕ to convince yourself that

$\cos\phi = \sqrt{1-\varsigma^2}.$

Putting it all together yields the result to be shown.

5.4 The logarithmic decrement

$$\delta = \log\left[x(t_k) / x(t_{k+1})\right]$$

is a measure of the damping. Show that

$$\varsigma = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}.$$

solution [1 point]

The times t_k are the times of the peaks in x. Although the presence of damping makes x a non-periodic function, the peaks occur regularly, with spacing $2\pi/\omega_{\mathrm{d}}$. Thus, $\omega_{\mathrm{d}}(t_{k+1}-t_k)=2\pi$, and

$$\delta = \log\left[x(t_k) / x(t_{k+1})\right] = \varsigma\omega_{\mathrm{n}}(t_{k+1} - t_k) = 2\pi\frac{\varsigma}{\sqrt{1-\varsigma^2}}.$$

Solving for ς yields the desired result.

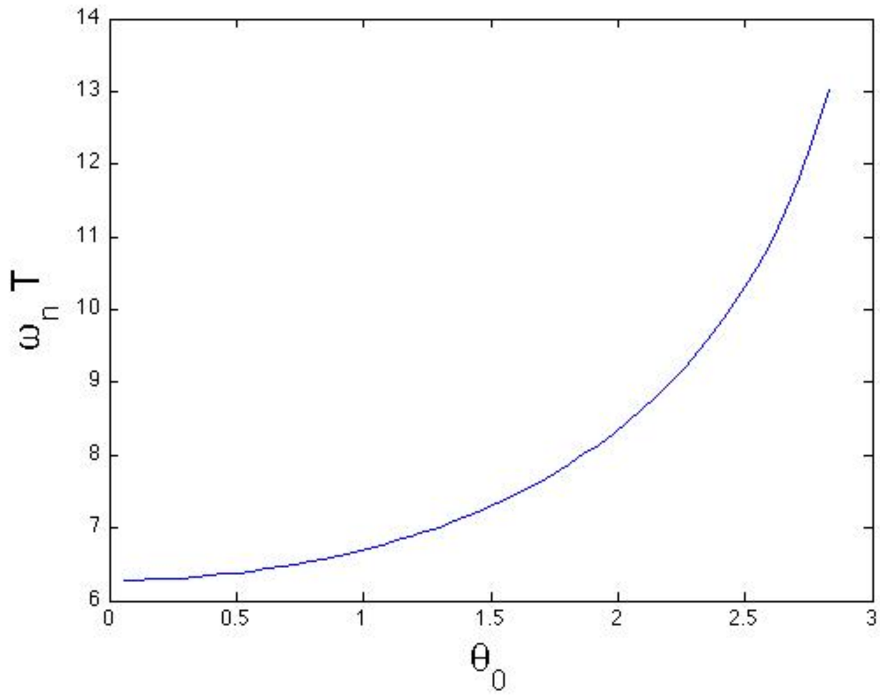
6. Using MATLAB, calculate how the period of an undamped pendulum depends on the initial amplitude θ_0 of the angle of the pendulum. Assume the pendulum starts from rest. Plot your answer for the period using appropriate dimensionless variables.

solution [5 points]

```
function  pendulumperiod
% calculate the period of a pendulum as a
% function of amplitude

npoints = 50; % number of amplitudes to
compute
initial_amp = zeros(npoints,1);
%preallocating
period = zeros(npoints,1);
for ii=1:npoints

    %be careful at the endpoints; theta0=0
would not
    %lead to an oscillation, and theta0=pi
would have
    %an infinite period!
    theta0=ii/(npoints)*0.9*pi;
    initial_amp(ii)=theta0;

    % time interval to solve the equations
    tstart = 0;
    tstop = 10*pi; %this has to be pretty
big as
                   %the initial theta gets
close to pi

    % initial value for [theta; thetadot]
    w_init = [theta0; 0];

    % solve
    options = odeset('Events',@events);
```

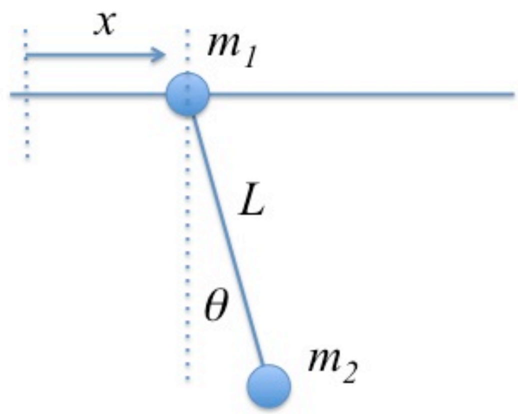
```
    [t_vals, sol_vals] = ...

ode45(@pend,[tstart,tstop],w_init,options);
    period(ii)=2*t_vals(end);
end

plot(initial_amp,period);
xlabel({'\theta_0'},'FontSize',18,'FontName
','Arial');
ylabel({'\omega_n
T'},'FontSize',18,'FontName','Arial');

    function dwds = pend(t,w)
        dwds = zeros(2,1);
        dwds(1)=w(2);
        dwds(2)=-sin(w(1));
    end;

    function
[eventvalue,stopthecalc,eventdirection]=eve
nts(t,w)
        eventvalue = w(2); %we'll just stop
it at half the period
        stopthecalc = 1;
        eventdirection = 1;
    end

end
```

The output is on the next page. Note that for small angles, we recover the independence of the period with respect to initial amplitude; the period is independent of the initial amplitude with errors of the order of the initial amplitude squared for small initial amplitude.



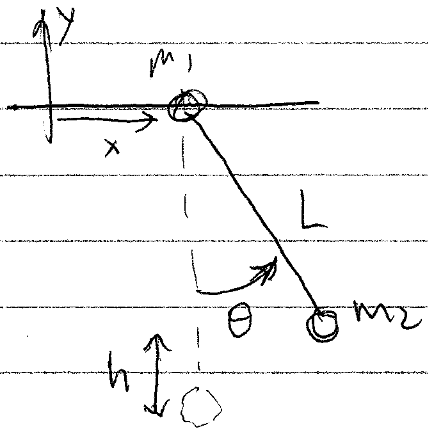
7. The figure below shows a pendulum in which the support point, with mass m_1, is free to slide back and forth without friction along a horizontal bar. The mass at the end of the pendulum is m_2. There is no initial velocity in the system, and there is no damping. The goal of this problem is to find the natural frequency.

7.1 Write down the total energy as a function of x and θ, and/or their time derivatives.
7.2 What principle relates $\dot{x}$ to θ and $\dot{\theta}$? Use this principle to find $\dot{x}$ in terms of θ and $\dot{\theta}$.
7.3 Write the total energy as a function of θ and $\dot{\theta}$.
7.4 Expand the energy for small angle. You need to go to second order in the small quantities.
7.5 Using the fact that the energy is constant for this conservative system, derive the equation of motion for small oscillations. What is the natural frequency?
7.6 Consider your answer in two different limits, $m_1 \gg m_2$, and $m_1 \ll m_2$. Can you explain why the natural frequency takes the value it does in these two limits?



x
m_1
L
θ
m_2

(7)



7.1 solution [1 point]

$$E = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_2 \dot{y}_2 + m_2 g y_2$$

$x_1 = x$ $\qquad$ $\dot{x}_1 = \dot{x}$

$x_2 = x + L\sin\theta$ $\qquad$ $\dot{x}_2 = \dot{x} + L\dot{\theta}\cos\theta$

$y_2 = -L\cos\theta$ $\qquad$ $\dot{y}_2 = L\dot{\theta}\sin\theta$

$$E = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 (\dot{x} + L\dot{\theta}\cos\theta)^2 + \frac{1}{2} m_2 L^2 \dot{\theta}^2 \sin^2\theta - m_2 g L\cos\theta$$

$$E = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2 + m_2 \dot{x} L\dot{\theta}\cos\theta + \frac{1}{2} m_2 L^2 \dot{\theta}^2 \cos^2\theta + \frac{1}{2} m_2 L^2 \dot{\theta}^2 \sin^2\theta - m_2 g L\cos\theta$$

$$E = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_2 \dot{x} L\dot{\theta}\cos\theta + \frac{1}{2} m_2 L^2 \dot{\theta}^2 - m_2 g L\cos\theta$$

7.2 Solution [1 point]

There are no x-forces on the two-mass system. Thus the x-component of momentum is conserved. Since the system starts from rest, $P_x = 0$.

Thus

$$m_1 \dot{x}_1 + m_2 \dot{x}_2 = 0$$

$$m_1 \dot{x} + m_2 (\dot{x} + L\dot{\theta}\cos\theta) = 0$$

$$\dot{x} = -\frac{m_2}{m_1+m_2} L\dot{\theta}\cos\theta$$

7.3 solution [1 point]

$$E = \frac{1}{2}\frac{m_2^3}{(m_1+m_2)} L^2\dot{\theta}^2\cos^2\theta - \frac{m_2^2}{m_1+m_2} L^2\dot{\theta}^2\cos^2\theta + \frac{1}{2}m_2 L^2\dot{\theta}^2 - m_2 g L\cos\theta$$

$$E = \frac{1}{2}m_2 L^2\dot{\theta}^2 - \frac{m_2^2}{2(m_1+m_2)} L^2\dot{\theta}^2\cos^2\theta - m_2 g L\cos\theta$$

7.4 solution [1 point]

expand in powers of θ - use $\cos\theta \simeq 1 - \theta^2/2 + \cdots$

$\cos^2\theta \simeq 1 - \theta^2 + \cdots$

already $O(\theta^2)$

$$E = \frac{1}{2}m_2 L^2\dot{\theta}^2 - \frac{m_2^2}{2(m_1+m_2)} L^2\dot{\theta}^2 - m_2 g L\left(1 - \frac{1}{2}\theta^2\right)$$

$$\boxed{E \simeq \frac{1}{2}\frac{m_1 m_2}{m_1+m_2} L^2\dot{\theta}^2 + \frac{m_2 g L\theta^2}{2} + \text{constant}}$$

Thus

$$m_1 \dot{x}_1 + m_2 \dot{x}_2 = 0$$

$$m_1 \dot{x} + m_2(\dot{x} + L\dot{\theta}\cos\theta) = 0$$

$$\dot{x} = -\frac{m_2}{m_1+m_2} L\dot{\theta}\cos\theta$$

7.3 solution [1 point]

$$E = \frac{1}{2}\frac{m_2^3}{(m_1+m_2)} L^2\dot{\theta}^2\cos^2\theta - \frac{m_2^2}{m_1+m_2} L^2\dot{\theta}^2\cos^2\theta + \frac{1}{2}m_2L^2\dot{\theta}^2 - m_2 gL\cos\theta$$

$$E = \frac{1}{2}m_2 L^2\dot{\theta}^2 - \frac{m_2^2}{2(m_1+m_2)} L^2\dot{\theta}^2\cos^2\theta - m_2 gL\cos\theta$$

7.4 solution [1 point]

expand in powers of θ - use $\cos\theta \simeq 1 - \theta^2/2 + \cdots$

$\cos^2\theta \simeq 1 - \theta^2 + \cdots$

already $O(\theta^2)$

$$E = \frac{1}{2}m_2L^2\dot{\theta}^2 - \frac{m_2^2}{2(m_1+m_2)} L^2\dot{\theta}^2 - m_2 gL\left(1 - \frac{1}{2}\theta^2\right)$$

$$\boxed{E \simeq \frac{1}{2}\frac{m_1 m_2}{m_1+m_2} L^2\dot{\theta}^2 + \frac{m_2 g L}{2}\theta^2 + \text{constant}}$$

Thus

$$m_1 \dot{x}_1 + m_2 \dot{x}_2 = 0$$

$$m_1 \dot{x} + m_2(\dot{x} + L\dot{\theta}\cos\theta) = 0$$

$$\dot{x} = -\frac{m_2}{m_1+m_2} L\dot{\theta}\cos\theta$$

7.3 solution [1 point]

$$E = \frac{1}{2}\frac{m_2^3}{(m_1+m_2)} L^2\dot{\theta}^2\cos^2\theta - \frac{m_2^2}{m_1+m_2} L^2\dot{\theta}^2\cos^2\theta + \frac{1}{2}m_2 L^2\dot{\theta}^2 - m_2 g L\cos\theta$$

$$E = \frac{1}{2}m_2 L^2\dot{\theta}^2 - \frac{m_2^2}{2(m_1+m_2)} L^2\dot{\theta}^2\cos^2\theta - m_2 g L\cos\theta$$

7.4 solution [1 point]

expand in powers of θ – use $\cos\theta \simeq 1 - \theta^2/2 + \dots$

$\cos^2\theta \simeq 1 - \theta^2 + \dots$

already $O(\theta^2)$

$$E = \frac{1}{2}m_2 L^2\dot{\theta}^2 - \frac{m_2^2}{2(m_1+m_2)} L^2\dot{\theta}^2 - m_2 g L\left(1 - \frac{1}{2}\theta^2\right)$$

$$\boxed{E \simeq \frac{1}{2}\frac{m_1 m_2}{m_1+m_2} L^2\dot{\theta}^2 + \frac{m_2 g L}{2}\theta^2 + \text{constant}}$$

7.5 solution [1 point]

$$\frac{dE}{dt}=0 \Rightarrow \frac{m_1 m_2}{m_1+m_2} L^2 \dot{\theta}\ddot{\theta} + m_2 g L \theta \dot{\theta} = 0$$

$$\boxed{\ddot{\theta} + \frac{g}{L}\frac{m_1+m_2}{m_1}\theta = 0}$$

comparing to $\ddot{\theta}+\omega_n^2\theta=0$, we see that the natural frequency is

$$\omega_n = \sqrt{\frac{g}{L}\frac{m_1+m_2}{m_1}}$$

7.6 solution [1 point]

When $m_1 \gg m_2$, the top mass hardly needs to move to ensure momentum conservation. So when $m_1 \gg m_2$, we expect to attain the case with stationary top point, for which $\omega_n = \sqrt{g/L}$. Our formula correctly shows this limit.

What about $m_2 \gg m_1$? ~~In~~ this limit, the mass #1 does all the moving — note $T \sim \frac{1}{2} m_1 L \dot{\theta}^2$ in this limit. But the potential energy is still controlled by m_2. So it is as if g is increased — ω_n increases proportionally to m_2/m_1 for m_2/m_1; $\omega_n \to \sqrt{\frac{g}{L}\frac{m_2}{m_1}}$