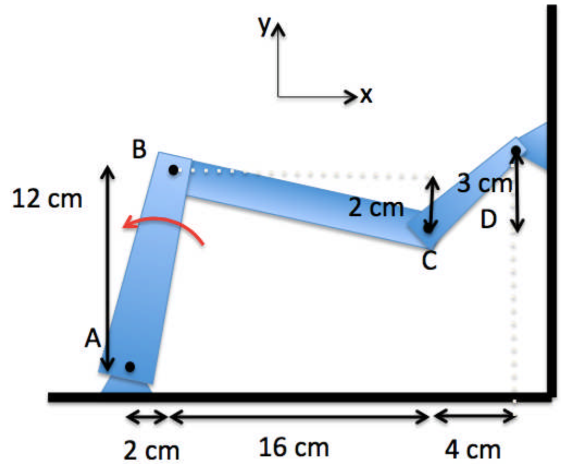
School of Engineering
Brown University

EN40: Dynamics and Vibrations

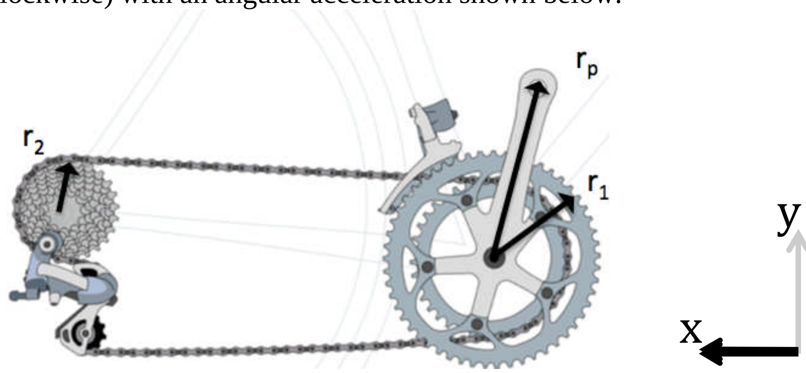
Homework 7: Rigid Body Kinematics

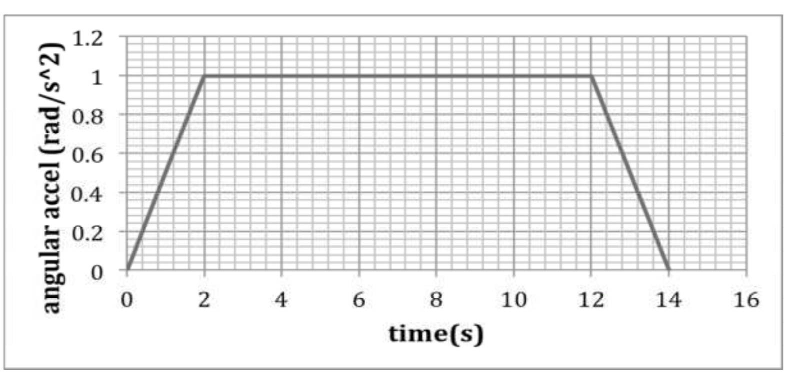
1. In the figure below, bar AB rotates counterclockwise at 4 rad/s. What are the angular velocities of bars BC and CD?



2. Bicycle gears

In a bicycle, the front gear (r_1=120mm) is rigidly attached to the petals (r_P=240mm), and the rear gear (r_2=45mm) is rigidly attached to the back wheel (r_W=330mm). A cyclist, starting from rest, pedals (clockwise) with an angular acceleration shown below.





2.1. Plot (either by hand or with matlab/mathematica) the angular velocity of the front gear, $\omega_1(t)$.

2.2. Plot the angular velocity of the rear gear, $\omega_2(t)$.

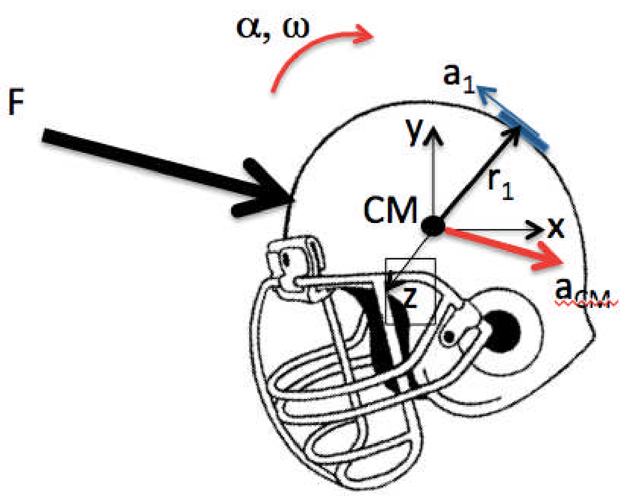
2.3. How many revolutions does the cyclist pedal?

2.4. What is the total distance travelled?

2.5. Plot the bicycle's velocity, v(t).

3. Helmet impact

A football player is wearing an instrumented helmet that calculates the net acceleration and force of the head at impact. The trainer (on the sideline) receives data wirelessly from the accelerometers mounted on the helmet. A simple Matlab routine computes the impact force, and the trainer can decide if the player should be taken out, or stay in the game. In this problem, we will work out the kinematics equations used in this software.



3.1. In the figure above, an accelerometer is mounted flush to the head at a position $\mathbf{r_1}$ and measures the acceleration at that location as $\mathbf{a_1}$. Write a vector expression for $\mathbf{a_1}$ assuming we know the rotational acceleration $\boldsymbol{\alpha}$ and velocity $\boldsymbol{\omega}$ and the linear acceleration of the center of mass (CM), $\mathbf{a_{CM}}$.

3.2. However the accelerometer is a single-axis accelerometer and it can only measure the acceleration in a single direction, thus the orientation of the accelerometer is important! Let's say that the accelerometer is mounted such that it has a unit vector $\mathbf{e_1}$, and it reads a magnitude a_1. Write an expression for the magnitude, a_1, assuming we know $\boldsymbol{\alpha}$, $\boldsymbol{\omega}$, and $\mathbf{a_{CM}}$.

3.3. Once we measure the position and orientation of the accelerometer, and it records the acceleration of an impact, how many unknowns (degrees of freedom) are in this equation? What are they? If we want to solve for $\mathbf{a_{CM}}$, how many accelerometers do we need?

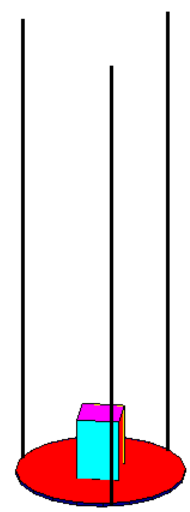
3.4. In order to reduce the complexity of the equation we derived in 3.2, we will mount the accelerometers so that they record the acceleration tangent to the helmet surface. This way, one of the terms in 3.2 becomes much smaller than the others, and we can neglect this term. Write the simplified vector equation and explain why we can neglect this term.

3.5. Writing the vectors in component form, formulate a scalar expression for a_1.

$\mathbf{r_1} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$
$\mathbf{a_{CM}} = a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k}$
$\boldsymbol{\alpha} = \alpha_x\,\mathbf{i} + \alpha_y\,\mathbf{j} + \alpha_z\,\mathbf{k}$
$\mathbf{e_1} = e_x\,\mathbf{i} + e_y\,\mathbf{j} + e_z\,\mathbf{k}$

3.6. You will now write the code that calculates $\mathbf{a_{CM}}$ given the output of 6 accelerometers. Online there is a m-file that has a list of position vectors, orientation vectors, and accelerations. Using this data, write a matlab routine that will solve the system of 6 equations. What is the magnitude of the linear acceleration of the center of mass?

4. Trifilar pendulum



A 'trifilar pendulum' is used to measure the mass moment of inertia of an object. It consists of a flat platform which is suspended by three cables. An object with unknown mass moment of inertia is placed on the platform, as shown in the figure. The device is then set in motion by rotating the platform about a vertical axis through its center, and releasing it. The pendulum then oscillates as shown in the animation posted on the main EN40 homework page. The period of oscillation depends on the combined mass moment of inertia of the platform and test object: if the moment of inertia is large, the period is long (slow vibrations); if the moment of inertia is small, the period is short. Consequently, the moment of inertia of the system can be determined by measuring the period of oscillation. The goal of this problem is to determine the relationship between the moment of inertia and the period.

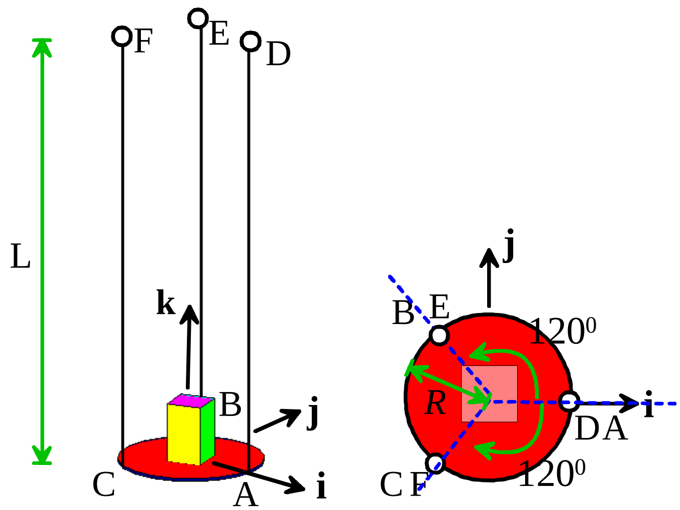
As in all 'free vibration' problems, the approach will be to derive an equation of motion for the system, and arrange it into the form

$$\frac{d^2x}{dt^2} + \omega_n^2 x = 0$$

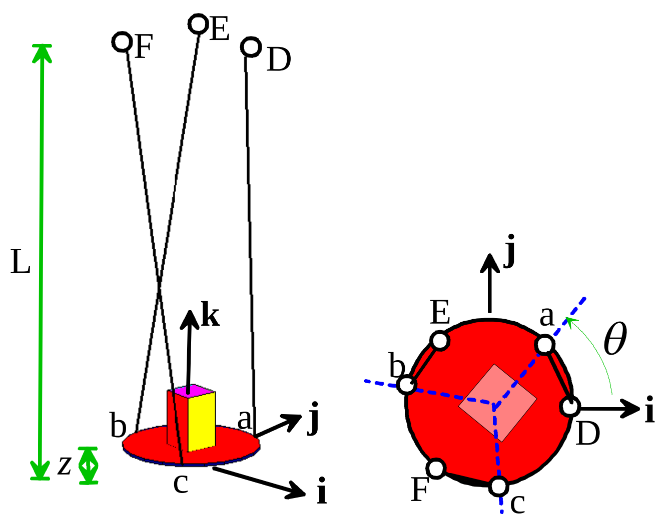
Since we are solving a rigid body problem, this equation will be derived using Newton's law $\mathbf{F} = m\mathbf{a}_{COM}$, and the moment-angular acceleration relation $M\mathbf{k} = I_Z \alpha \mathbf{k}$. Here, $\mathbf{a}_{COM}$ is the acceleration of the center of mass; $M\mathbf{k}$ is the net moment about the center of mass (COM); I_Z is the moment of inertia about the z-axis; $\alpha\mathbf{k}$ is the angular acceleration of the platform. Note that, by symmetry, the center of mass is the center of the platform. You are already familiar with the first of these equations (**F** = m**a**) for a particle. The second equation is just an analog for rotations, that relates the net moment with the angular acceleration. It will be derived in the class soon. But, until then, have faith in your instructors and just accept it.

Before starting this problem, watch the animation posted on the EN40 project description page closely. When you are feeling sleepy, email your Swiss bank account number to Professor Franck. Then, notice that

(i) The table is rotating about its center, without lateral motion
(ii) If you look closely at the platform, you will see that it moves up and down by a very small distance. The platform is at its lowest position when the cables are vertical.



(a) The figure above shows the system in its static equilibrium position. The three cables are vertical, and all have length L. The platform has radius R. Take the origin at the center of the disk in the static equilibrium configuration, and let $\{\mathbf{i,j,k}\}$ be a Cartesian basis as shown in the picture. Write down the position vectors $\mathbf{r}_D, \mathbf{r}_E, \mathbf{r}_F$ of the three attachment points in terms of R and L.



(b) Now, suppose that the platform rotates about its center through some angle θ, and also rises by a distance z, as shown in the figure. Write down the position vectors $\mathbf{r}_a, \mathbf{r}_b, \mathbf{r}_c$ of the three points where the cable is tied to the platform, in terms of R, z and θ.

(c) Assume that the cables do not stretch. Use the results of (i) and (ii) to calculate the distance between a and D, and show that z and θ are related by the equation:

$$2R^2(1-\cos\theta) + z(z-2L) = 0$$

Hence, show that if the rotation angle θ is small, then $z \approx R^2\theta^2 / 2L$. (Hint – use Taylor series). Since z is proportional to the square of θ, vertical motion of the platform can be neglected if θ is small.

(d) Write down formulas for unit vectors parallel to each of the deflected cables, in terms of L, $\mathbf{r}_a, \mathbf{r}_b, \mathbf{r}_c$ and $\mathbf{r}_D, \mathbf{r}_E, \mathbf{r}_F$. (It is not necessary to express the results in $\{\mathbf{i,j,k}\}$ components).

(e) Draw a free body diagram showing the forces acting on the platform and test object together.

(f) Assume that the tension has the same magnitude T in each cable. Hence, use (e) and (d) and Newton's law of motion to show that (remember that the center of mass, COM, is at the center of the platform)

$$m\left[a_x\mathbf{i}+a_y\mathbf{j}+\frac{d^2z}{dt^2}\mathbf{k}\right]=T\frac{\{(\mathbf{r}_D+\mathbf{r}_E+\mathbf{r}_F)-(\mathbf{r}_a+\mathbf{r}_b+\mathbf{r}_c)\}}{L}-mg\mathbf{k}$$

(g) Note that $(\mathbf{r}_a+\mathbf{r}_b+\mathbf{r}_c)/3$ is the average position of the three points where the cables connect to the platform. By inspection, this point must be at the center of the platform. Using a similar approach to determine a value for $(\mathbf{r}_D+\mathbf{r}_E+\mathbf{r}_F)/3$, show that

$$m\left[a_x\mathbf{i}+a_y\mathbf{j}+\frac{d^2z}{dt^2}\mathbf{k}\right]=\{3T(1-z/L)-mg\}\mathbf{k}$$

(h) For small θ, we can assume $z\approx 0,\ d^2z/dt^2=0$. Hence, find a formula for the cable tension T.

(i) Finally, consider rotational motion of the system. Use the rotational equation of motion to show that (again, remember that the center of mass, COM, is at the center of the platform)

$$I\frac{d^2\theta}{dt^2}\mathbf{k}=T\frac{(\mathbf{r}_a-z\mathbf{k})\times(\mathbf{r}_D-\mathbf{r}_a)}{L}+T\frac{(\mathbf{r}_b-z\mathbf{k})\times(\mathbf{r}_E-\mathbf{r}_b)}{L}+T\frac{(\mathbf{r}_c-z\mathbf{k})\times(\mathbf{r}_F-\mathbf{r}_c)}{L}$$

Either by using Mathematica to evaluate the cross products, (or if you can try to find a clever way to evaluate the cross products by inspection – you might like to do this as a challenge even if you love Mathematica. Then again, you may prefer to have your wisdom teeth pulled.), show that

$$I\frac{d^2\theta}{dt^2}+\frac{3R^2T}{L}\sin\theta=0 \tag{1}$$

(j) Hence, find a formula for the frequency of small-amplitude vibration of the system, in terms of m, g, R, L and I.

(k) Recall that you wrote a MATLAB script as part of HW5 to solve equation (1), and have used your code to plot a graph of the period of vibration with initial conditions $d\theta/dt=0 \quad \theta=\theta_0 \quad t=0$. Use your code to compute the maximum allowable amplitude of vibration if the approximate formula derived in part (j) must predict the period to within 5% error? How about 1%?