HW 6 solutions ENGN 0040 2012

[2 points] ① steady-state motion of an oscillator with damping, and driving $F = F_0 \sin\omega t$.

The governing equation is $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m}\sin\omega t$

But sine is the same as cosine with a different choice for origin: $\sin\omega t = \cos(\omega t - \pi/2) = \cos\omega t'$, where $t' = t - \frac{\pi}{2\omega}$.

Note that $\frac{d}{dt'} = \frac{d}{dt}$. Thus,

$$\frac{d^2x}{dt'^2} + 2\zeta\omega_n\frac{dx}{dt'} + \omega_n^2 x = \frac{F_0}{m}\cos\omega t'$$

We can read off the solution from the class notes -

$$x = A\cos(\omega t' + \phi), \quad \text{with} \quad \boxed{A = \frac{F_0/m}{[(\omega_n^2-\omega^2)^2 + 4\zeta^2\omega^2\omega_n^2]^{1/2}}}$$

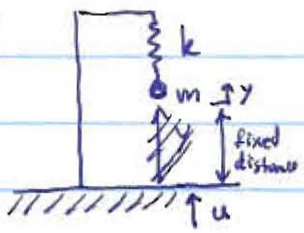
and $\boxed{\tan\phi = \frac{-2\zeta\omega\omega_n}{\omega_n^2-\omega^2}}$

reverting to t, we have $x = A\cos(\omega t - \frac{\pi}{2} + \phi)$

$$\boxed{x = A\sin(\omega t + \phi)}$$

Alternatively, we could solve the problem directly, using $\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = -i\frac{F_0}{m}e^{i\omega t}$, with $x = \mathrm{Re}\, z$, and $z = A_s e^{i(\omega t + \phi_s)}$. Following the same steps as in lecture, I find $\boxed{A_s = A}$ and $\tan\phi_s = \frac{\omega_n^2 - \omega^2}{2\zeta\omega_n\omega}$. Note that $\tan\phi_s$ has period π - we must have $\phi_s \to 0$ when $\omega \to 0$, so $\boxed{\phi_s = \phi - \frac{\pi}{2}}$

(2)



The earth shakes with displacement u.
y is the coordinate of the mass relative to the earth – we take out a constant offset y_0, so that y is a deflection.

[1 point] 2.1 $F = m(\ddot{u}+\ddot{y})$ since $a = \ddot{u}+\ddot{y}$

The spring force is $-ky$

The damper give force $-b\dot{y}$

$$\therefore \quad -ky - b\dot{y} = m(\ddot{u}+\ddot{y})$$

$$m\ddot{y} + b\dot{y} + ky = -m\ddot{u}$$

$$\boxed{\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = -\ddot{u}}$$

[1 point] 2.2 $u = u_0 \cos\omega t$

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = -\omega^2 u_0 \cos\omega t \Rightarrow y = A\cos(\omega t + \phi)$$

with

$$A = \frac{-\omega^2 u_0}{\left[(\omega_n^2-\omega^2)^2 + 4\zeta\omega^2\omega_n^2\right]^{1/2}} \qquad \tan\phi = -\frac{2\zeta\omega\omega_n}{\omega_n^2-\omega^2}$$

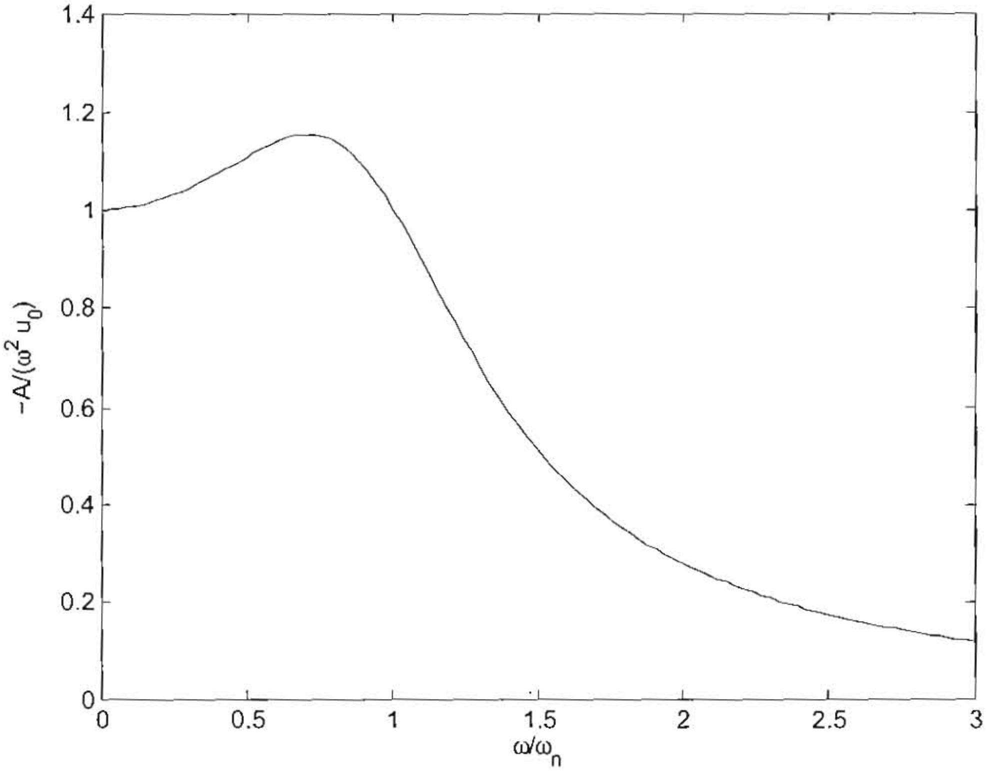
[1 point] 2.3 is on the next page

[1 point] 2.4 $2\pi/\omega_n \simeq 30\ \mathrm{s}$ $\quad Q \simeq 2$

$\omega^2 u_0 \simeq 10^{-2}\ \mathrm{m/s}$ $\qquad Q = \omega_n/\gamma = 1/2\zeta \Rightarrow \zeta = 1/4$

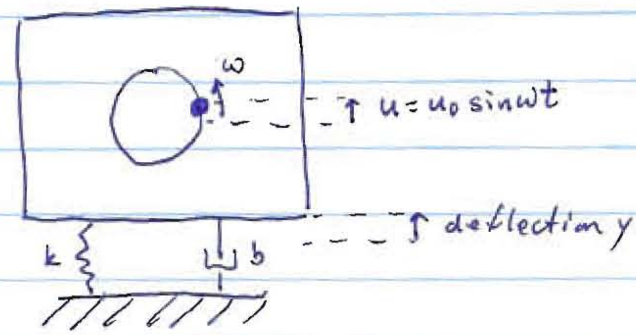
$\omega = 2\pi f = 2\pi \cdot 10\ \mathrm{rad/s}$

$$|A| = \frac{+10^{-2}\ \mathrm{m}}{\left\{\left[\left(\frac{2\pi}{30}\right)^2 - (2\pi)^2 10^2\right]^2 + \frac{4}{4}\frac{(2\pi)^2 10^2 (2\pi)^2}{30^2}\right\}^{1/2}} \simeq 2.5\ \mu\mathrm{m}$$

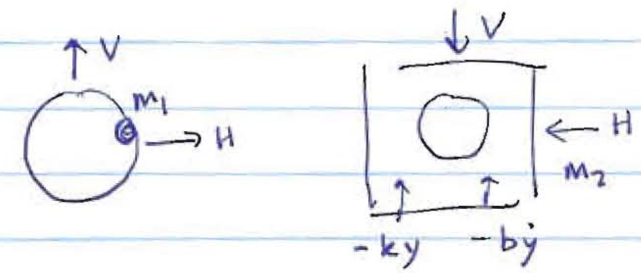


1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
-A/(ω² u₀)
ω/ωₙ

(3)



[1 point] 3.1



[1 point] 3.2 The forces acting on the drum-ball system are the reaction forces H and V. The forces acting on the rest of the washing machine are −H, −V, and the spring and damper forces.

[1 point] 3.3

$$m_1(\ddot{y}+\ddot{u}) = V$$

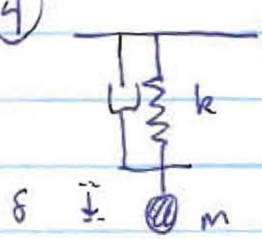
$$m_2\ddot{y} = -V - ky - b\dot{y}$$

$$= -m_1(\ddot{y}+\ddot{u}) - ky - b\dot{y}$$

$$(m_1+m_2)\ddot{y} + b\dot{y} + ky = -m_1\ddot{u}$$

$$\boxed{\ddot{y} + \frac{b}{m_1+m_2}\dot{y} + \frac{k}{m_1+m_2}y = +\frac{m_1\omega^2}{m_1+m_2}\sin\omega t}$$

4



$u = u_0 \cos\omega t$

$u_0 = 1\,mm$ $\quad Q = 15$

$m = 2\,kg$

$\delta = 2.5\,cm$

[1 point] 4.1 $\omega_n = \sqrt{\frac{k}{m}}$. In equilibrium, $k\delta = mg$ so $k = \frac{mg}{\delta}$

Thus, $\omega_n = \sqrt{g/\delta} = \sqrt{\frac{9.81\,m/s^2}{0.025\,m}} \simeq 19.8\,rad/s$

[1 point] 4.2 [The quality factor Q is pretty high, ~~so~~ so $\omega_n \simeq \omega_{max}$.]

$$A = Q\,A(\omega=0)$$
$$= (15)(2.5\,cm)$$
$$A = 37.5\,cm$$

[1 point] 4.3 Mean power input

from lecture,

$$\langle P(\omega)\rangle = \frac{1}{2Q}\frac{F_0^2}{m}\omega_n \frac{1}{\left(\frac{\omega_n}{\omega}-\omega\right)^2 + \omega_n^2/Q^2}$$

for $\ddot{x} + 2\zeta\dot{x}\omega_n + \omega_n^2 x = F_0/m \cos\omega t$, we must get the equation of motion for this problem into this form. The acceleration of the mass relative to the lab frame is $\ddot{u} + \ddot{y}$.

Thus $\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = -\ddot{u} = \omega^2 u_0 \cos\omega t$

$$\Rightarrow \frac{F_0}{m} = \omega^2 u_0$$

thus

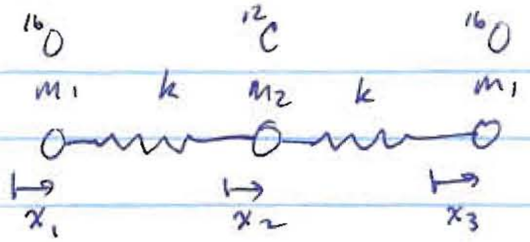
$$\langle P(\omega)\rangle = \frac{1}{2Q}\, m\,\omega^4 u_0^2 \,\frac{\omega_n}{\left(\frac{\omega_n^2}{\omega}-\omega\right)^2 + \omega_n^2/Q^2}$$

$$\omega = \omega_n\, 1.02$$

$$\langle P(1.02\omega_n)\rangle = \frac{1}{2\cdot 15}(2\,\mathrm{kg})\,\frac{\left[(1.02)(19.8)\right]^4 s^{-4}\, 10^{-6}\, m^2\, (19.8\, s^{-1})}{\left(\frac{1}{1.02}-1.02\right)^2 (19.8\, s^{-1})^2 + \frac{(19.8\, s^{-1})^2}{15^2}}$$

$$\simeq 93\ \mathrm{kW}$$

5.



[1 point] 5.1 equations of motion for each particle

$\Sigma F_x = m\ddot{x}$ for each particle

$$\boxed{\begin{aligned} k(x_2 - x_1) &= m_1 \ddot{x}_1 \\ k(x_3 - x_2) - k(x_2 - x_1) &= m_2 \ddot{x}_2 \\ -k(x_3 - x_2) &= m_1 \ddot{x}_3 \end{aligned}}$$

[1 point] 5.2 the center of mass stays fixed if the molecule is initially at rest.

Thus $m_1 x_1 + m_2 x_2 + m_1 x_3 = 0$

$$\boxed{x_2 = -\frac{m_1}{m_2}(x_1 + x_3)}$$

$\therefore$ If we know the motion of x_1 and x_3, we know the motion of x_2.

[1 point] 5.3 $$m_1 \ddot{x}_1 = k\left[-\frac{m_1}{m_2}(x_1 + x_3) - x_1\right]$$

$$m_1 \ddot{x}_1 = -k\left[\frac{m_1 + m_2}{m_2} x_1 + \frac{m_1}{m_2} x_3\right]$$

$$\ddot{x}_1 + \frac{k}{m_1}\left[\frac{m_1 + m_2}{m_2} x_1 + \frac{m_1}{m_2} x_3\right] = 0$$

$$m_1 \ddot{x}_3 = -k\left(x_3 + \frac{m_1}{m_2} x_1 + \frac{m_1}{m_2} x_3\right)$$

$$m_1 \ddot{x}_3 = -k\left(\frac{m_1 + m_2}{m_2} x_3 + \frac{m_1}{m_2} x_1\right) \quad \text{or} \quad \ddot{x}_3 + \frac{k}{m_1}\left(\frac{m_1 + m_2}{m_2} x_3 + \frac{m_1}{m_2} x_1\right) = 0$$

Thus

$$\boxed{\begin{aligned} &\ddot{x}_1 + \frac{k}{m_1}\left[\frac{m_1+m_2}{m_2}x_1 + \frac{m_1}{m_2}x_3\right] = 0 \\ &\ddot{x}_3 + \frac{k}{m_1}\left[\frac{m_1+m_2}{m_2}x_3 + \frac{m_1}{m_2}x_1\right] = 0 \end{aligned}}$$

[1 point] 5.4 sum

$$\ddot{x}_1 + \ddot{x}_3 + \frac{k}{m_1}\left[\frac{m_1+m_2}{m_2}(x_1+x_3) + \frac{m_1}{m_2}(x_1+x_3)\right] = 0$$

let $q_2 = x_1 + x_3$

then $$\ddot{q}_2 + \frac{k}{m_1}\left(\frac{m_1+m_2}{m_2} + \frac{m_1}{m_2}\right) q_2 = 0$$

or $$\boxed{\ddot{q}_2 + \frac{k}{m_1}\frac{m_2+2m_1}{m_2} q_2 = 0} \Rightarrow q_2 = D\cos\omega_2 t$$

$$\boxed{\omega_2^2 = \frac{k}{m_1}\frac{m_2+2m_1}{m_2}}$$

difference

$$\ddot{x}_1 - \ddot{x}_3 + \frac{k}{m_1}\left[\frac{m_1+m_2}{m_2}(x_1-x_3) - \frac{m_1}{m_2}(x_1-x_3)\right] = 0$$

let $q_{\ast} = x_1 - x_3$

$$\ddot{q}_1 + \frac{k}{m_1}\frac{m_2}{m_2} q_1 = 0$$

or $$\boxed{\ddot{q}_1 + \frac{k}{m_1} q_1 = 0} \Rightarrow q_1 = C\cos\omega_1 t$$

$$\boxed{\omega_1^2 = k/m_1}$$

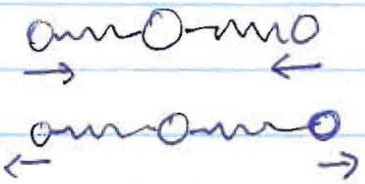
[1 point] 5.5 The normal modes

(1) symmetric mode − $q_1 = C\cos\omega_1 t$

$q_2 = 0 \Rightarrow x_1 = -x_3$

$t = 0$

$t = \frac{\pi}{\omega_1}$

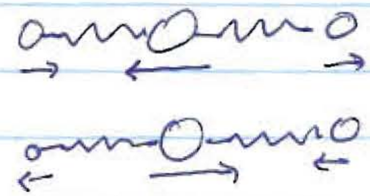


carbon atom does not move in this mode

(2) asymmetric mode $\quad q_1 = 0 \Rightarrow x_1 = x_3$

$t = 0$

$t = \frac{\pi}{\omega_2}$



$\dot{x}_2 = -\frac{m_1}{m_2} \cdot 2\dot{x}_1$ for $x_1 = x_3$

$= -\frac{16}{12} \cdot 2\dot{x}_1 = -\frac{8}{3}\dot{x}_1$

[1 point] 5.6

$$\left.\begin{aligned} q_1 &= x_1 - x_3 \\ q_2 &= x_1 + x_3 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 &= \tfrac{1}{2}(q_1 + q_2) \\ x_3 &= \tfrac{1}{2}(q_2 - q_1) \end{aligned}$$

$$x_1 = \frac{C}{2}\cos\omega_1 t + \frac{D}{2}\cos\omega_2 t$$

$$x_3 = \frac{D}{2}\cos\omega_1 t - \frac{C}{2}\cos\omega_2 t$$

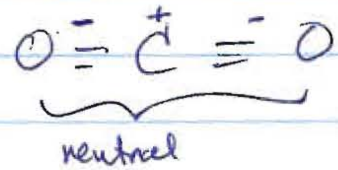
if $x_1(0) = X_0,\ x_2(0) = 0$, then

$$\boxed{\begin{aligned} x_1 &= \frac{X_0}{2}(\cos\omega_1 t + \cos\omega_2 t) \\ x_2 &= \frac{X_0}{2}(\cos\omega_1 t - \cos\omega_2 t) \end{aligned}}$$

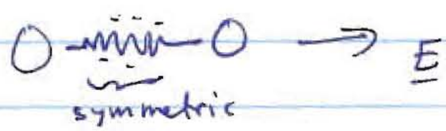
5.7 The electrons are shared equally between the atoms in O_2 and N_2 because of symmetry: $N \equiv N$.

↓

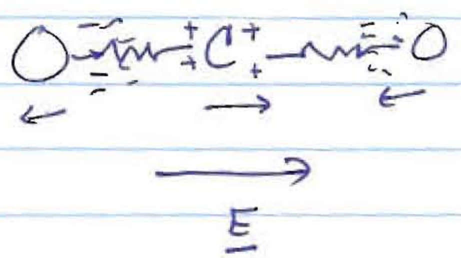
Oxygen is strongly electronegative, so there is more negative charge closer ~~than~~ to the oxygens:



Imagine an N_2 or O_2 molecule in an electric field. There is no tendency for the electric field to start the molecule vibrating

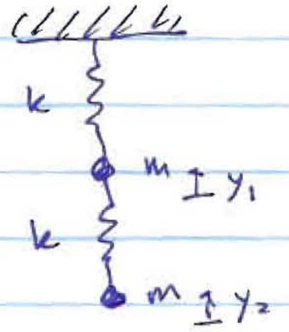


But for CO_2, the carbon will be pulled one way, and the oxygens will be pulled the other way



Thus, an electric field <u>can</u> excite the mode q_1. It turns out the excitation energy for this mode corresponds to light with a wavelength in the infrared. CO_2 molecules can absorb IR radiation - O_2 and N_2 cannot

6



[3 points] 6.1 first get the equations of motion

$$m\ddot{y}_1 = -ky_1 + k(y_2 - y_1)$$
$$m\ddot{y}_2 = -k(y_2 - y_1)$$

or
$$\ddot{y}_1 + \omega_0^2 y_1 - \omega_0^2(y_2 - y_1) = 0$$
$$\ddot{y}_2 + \omega_0^2(y_2 - y_1) = 0$$

or
$$\ddot{y}_1 + 2\omega_0^2 y_1 - \omega_0^2 y_2 = 0$$
$$\ddot{y}_2 - \omega_0^2 y_1 + \omega_0^2 y_2 = 0$$

for a normal mode, both masses oscillate at the same frequency:

try
$$y_1 = C \cos\omega t$$
$$y_2 = D \cos\omega t$$

$$-C\omega^2 + 2\omega_0^2 C - \omega_0^2 D = 0$$
$$-D\omega^2 - C\omega_0^2 + D\omega_0^2 = 0$$

or
$$\left.\begin{aligned} C(2\omega_0^2 - \omega^2) - \omega_0^2 D &= 0 \\ -\omega_0^2 C + D(\omega_0^2 - \omega^2) &= 0 \end{aligned}\right\} \text{two equations for } \underline{\underline{\text{three}}} \text{ unknowns } C, D, \text{ and } \omega$$

If we say these two equations are true for all values of ω, then we must conclude $C = D = 0$. This is not a normal mode. Instead, we find the value of ω for which these two equations become dependent (multiples of each other).

This amounts to solving for C/D, and demanding that both equations give the same result for C/D.

Thus
$$\frac{C}{D} = \frac{\omega_0^2}{2\omega_0^2 - \omega^2}$$
$$\frac{C}{D} = \frac{\omega_0^2 - \omega^2}{\omega_0^2}$$

Both equations must give the same value of C/D, so equate them to find ω:
$$\frac{\omega_0^2}{2\omega_0^2 - \omega^2} = \frac{\omega_0^2 - \omega^2}{\omega_0^2}$$
$$(2\omega_0^2 - \omega^2)(\omega_0^2 - \omega^2) - \omega_0^4 = 0$$
$$\omega^4 - 3\omega_0^2\omega^2 + \omega_0^4 = 0$$
$$\Rightarrow \boxed{\frac{\omega^2}{\omega_0^2} = \frac{3 \pm \sqrt{5}}{2}}$$

Also Note $\frac{\omega}{\omega_0} = \sqrt{\frac{3 \pm \sqrt{5}}{2}} = \frac{1 \pm \sqrt{5}}{2}$

$$\boxed{\frac{\omega}{\omega_0} = \frac{1 \pm \sqrt{5}}{2}}$$

so we have two natural frequencies,

$$\omega_\pm = \sqrt{\frac{k}{m}}\ \frac{1 \pm \sqrt{5}}{2}$$

[2 points]

6.2 there are two normal modes –

$$y_1 = C\cos\omega_- t \qquad \text{and} \qquad y_1 = C'\cos\omega_+ t$$
$$y_2 = D\cos\omega_- t \qquad\qquad\qquad y_2 = D'\cos\omega_+ t$$

whole system oscillates with frequency ω_-

whole system oscillates with frequency ω_+

for the normal mode with frequency ω_-, the ratio of amplitude of the motions of the masses is

$$\left(\frac{C}{D}\right)_- = \frac{\omega_0^2}{2\omega_0^2 - \omega_-^2} = \frac{\omega_0^2}{2\omega_0^2 - \omega_0^2\,\frac{3-\sqrt{5}}{2}} = \frac{1}{\frac{1}{2} + \sqrt{5}/2}$$

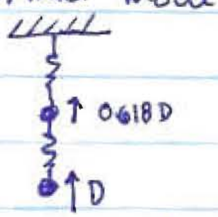
$$= \frac{2}{1+\sqrt{5}}\,\frac{1-\sqrt{5}}{1-\sqrt{5}} = \frac{2(1-\sqrt{5})}{1-5} = \boxed{\frac{-1+\sqrt{5}}{2} = \left(\frac{C}{D}\right)_-}$$

Likewise $\left(\frac{C}{D}\right)_+ = \frac{\omega_0^2}{2\omega_0^2 - \omega_+^2} = \frac{\omega_0^2}{2\omega_0^2 - \omega_0^2\,\frac{3+\sqrt{5}}{2}} = \frac{1}{\frac{1}{2} - \frac{\sqrt{5}}{2}} = \frac{2(1+\sqrt{5})}{1-5}$

$$\boxed{\left(\frac{C}{D}\right)_+ = -\frac{(1+\sqrt{5})}{2}}$$

First normal mode

$$\boxed{\begin{aligned} y_1 &= \frac{\sqrt{5}-1}{2} D\cos\omega_- t \\ y_2 &= D\cos\omega_- t \end{aligned}}$$



Second normal mode

$$\boxed{\begin{aligned} y_1 &= -\frac{1+\sqrt{5}}{2} D\cos\omega_+ t \\ y_2 &= D\cos\omega_+ t \end{aligned}}$$