School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 6: Forced Vibrations
Due Friday April 5th

1. Springs are used for many different engineering and science applications. Typical shock absorbers for passenger cars have the spring constant approximately 10 to a few hundred kN/m while a physics laboratory slinky has its typical spring constant in the range of 0.1 – 1.0 N/m. As shown in the following chart, the contact-mode atomic force microscope (AFM) cantilever has very soft spring constant which is close to a very soft slinky.

In AFM community, researchers use effective mass m_{eff} to model the vibration of the AFM cantilever for the first mode natural frequency (or resonant frequency) ω_{n1} to have $\omega_{\mathrm{n1}} = \sqrt{k/m_{\mathrm{eff}}}$, where k is the spring constant of the cantilever.

Find the effective masses for the three different AFM cantilevers listed below. Use (300kHz, 48N/m) for the short and (190kHz, 40N/m) for the long cantilevers of Dynamic mode / tapping mode AFM, and (28kHz, 0.2N/m) for the short and (12kHz, 0.1N/m) for the long cantilevers of the contact mode AFM. Compare the effective masses to the real mass of the cantilever (density times the volume). The cantilever is made of silicon and the density is $2.65\, g/cm^3$.

Standard cantilever types for common applications:

Dynamic mode / tapping mode AFM Available as a long (225 µm) or short (125 µm) cantilever	Length: Typically 125 - 225 µm Width: Typically 40 µm Thickness: 4 - 8 µm Resonant Freq: 190 - 300 KHz Spring Constant: 40 - 48 N/m
Force Modulation mode AFM	Length: Typically 225 µm Width: Typically 45µm Thickness: 2.5 µm Resonant Freq: 60 KHz Spring Constant: 3 N/m
Contact mode AFM Available as a long (450 µm) or short (225 µm) cantilever	Length: Typically 225 - 450µm Width: Typically 28-40 µm Thickness: 1- 2 µm Resonant Freq: 12 - 28 KHz Spring Constant: 0.1 - 0.2 N/m

Solution: [Both sets are correct for grading: 1 point each for the total of 5 points]

$$m_{\text{eff}} = k/\omega_{\text{n1}}^2$$

Answer based on original data set:

	k(N/m)	f(kHz)	w_n1(rad/s)	m_eff(kg)	length(m)	width(m)	thickness(m)	density(kg/m^3)	mass(kg)	m_eff/m
DT-AFM(long)	40	190	1193804.96	2.80668E-11	0.000225	0.00004	0.000004	2650	9.54E-11	0.2942
DT-AFM(short)	48	300	1884955.2	1.35095E-11	0.000125	0.00004	0.000008	2650	1.06E-10	0.12745
FM-AFM	3	60	376991.04	2.11086E-11	0.000225	0.000045	0.0000025	2650	6.71E-11	0.31469
CM-AFM(long)	0.1	12	75398.208	1.75905E-11	0.00045	0.000028	0.000001	2650	3.34E-11	0.52682
CM-AFM(short)	0.2	28	175929.152	6.46181E-12	0.000225	0.00004	0.000002	2650	4.77E-11	0.13547

Answer based on revised data set:

	k(N/m)	f(kHz)	w_n1(rad/s)	m_eff(kg)	length(m)	width(m)	thickness(m)	density(kg/m^3)	mass(kg)	m_eff/m
DT-AFM(long)	40	190	1193804.96	2.80668E-11	0.000225	0.00004	0.000004	2650	9.54E-11	0.2942
DT-AFM(short)	48	300	1884955.2	1.35095E-11	0.000125	0.00004	0.000004	2650	5.3E-11	0.2549
FM-AFM	3	60	376991.04	2.11086E-11	0.000225	0.000045	0.0000025	2650	6.71E-11	0.31469
CM-AFM(long)	0.1	12	75398.208	1.75905E-11	0.00045	0.000028	0.000002	2650	6.68E-11	0.26341
CM-AFM(short)	0.2	28	175929.152	6.46181E-12	0.000225	0.00004	0.000001	2650	2.39E-11	0.27094

2. A passenger car has front and rear suspensions composed of identical four spring dashpot shock absorbers, one for each wheel. The weight of the car is 1900kg and the initial sag (shortening of the spring by the weight) of the shock absorber is 14cm.

(1) Find the spring constant of the shock absorber.

Solution:

$$k = \left\{1900(\text{kg}) \times 9.8(\text{m/sec}^2)/4\right\} / 0.14(\text{m}) / 1000 = 33(\text{kN/m})$$ **[2 points]**

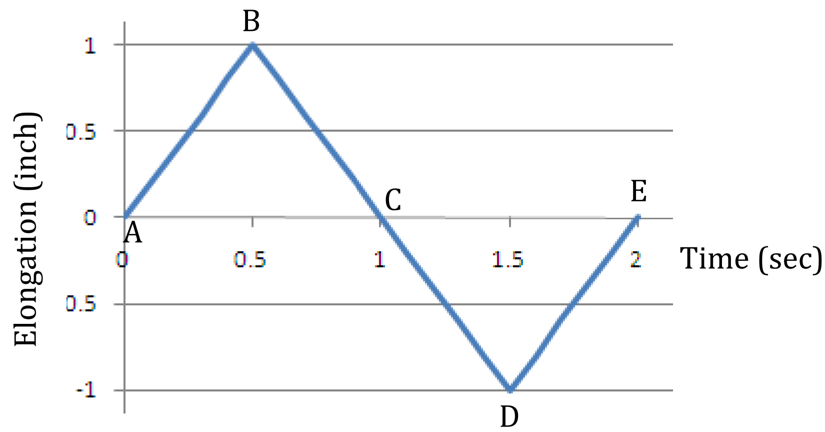
(2) In lecture 4, we defined the <u>damping ratio</u> $\zeta = \lambda / \left(2\sqrt{km}\right)$, where λ is the <u>damping coefficient</u> which is measured in a unit of kg/s and sometimes denoted by c in other text books. It was defined that the damping is critical if the damping ratio $\zeta = 1$. For most of passenger cars the shock absorbers are designed under-damped to have a typical value of $\zeta = 0.56$. Find the damping coefficient of this passenger car.

Solution: **[2 points]**

$$\lambda = 2\zeta\sqrt{km} = 2 \times 0.56\sqrt{33000(\text{kg/s}^2) \times (1900/4)(\text{kg})} = 4400(\text{kg/s})$$

3. Here we will see the energy absorption characteristics of the above shock absorber.

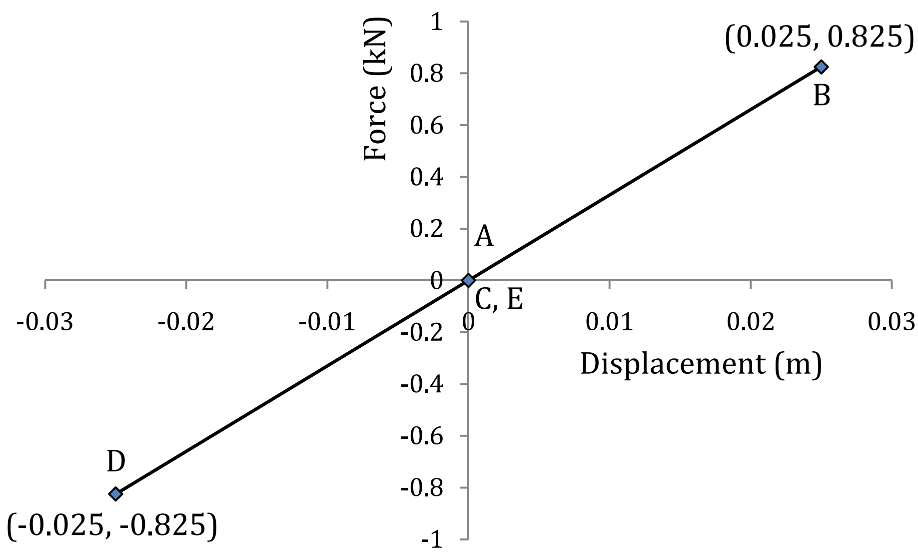
(1) Consider an elongation displacement cycle of the shock absorber as shown below.



(1.1) Draw the force - displacement diagram for the force applied to the spring during the displacement cycle. Mark the corresponding points of A, B, C, D and E on the drawing. What is the work dissipated during this cycle?

Solution:

Drawing [2 points]



Work dissipated = 0. **[2 points]**

(1.2) Draw the force - displacement diagram for the force applied to the dashpot during the displacement cycle. Mark the corresponding points of A, B, C, D and E on the drawing. What is the work dissipated during this cycle?

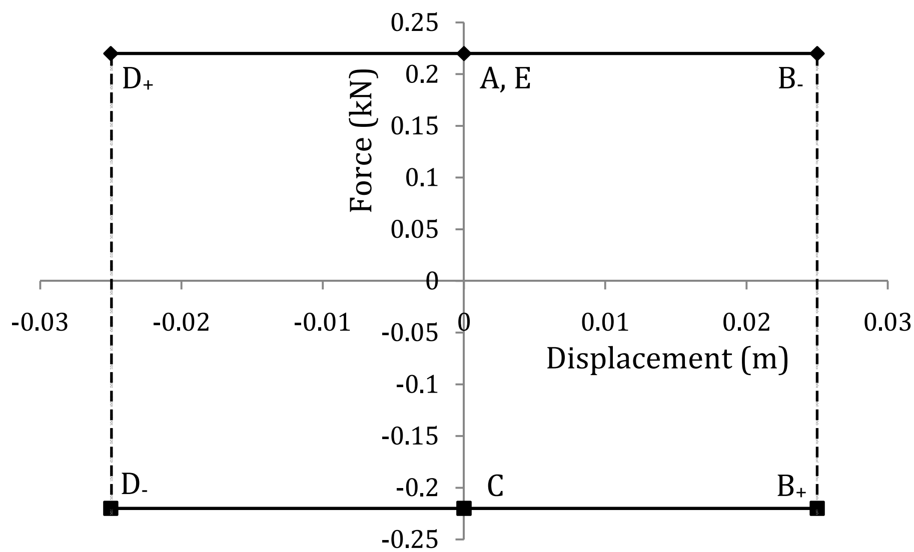
Solution:

Drawing [2 points]

$F = \lambda \times \dot{x} = 4400(\mathrm{kg/s}) \times 0.05(\mathrm{m/s})/1000 = 0.22(\mathrm{kN})$ for A – B.

$F = \lambda \times \dot{x} = 4400(\mathrm{kg/s}) \times (-0.05)(\mathrm{m/s})/1000 = -0.22(\mathrm{kN})$ for B – C – D.

$F = \lambda \times \dot{x} = 4400(\mathrm{kg/s}) \times 0.05(\mathrm{m/s})/1000 = 0.22(\mathrm{kN})$ for D – E.

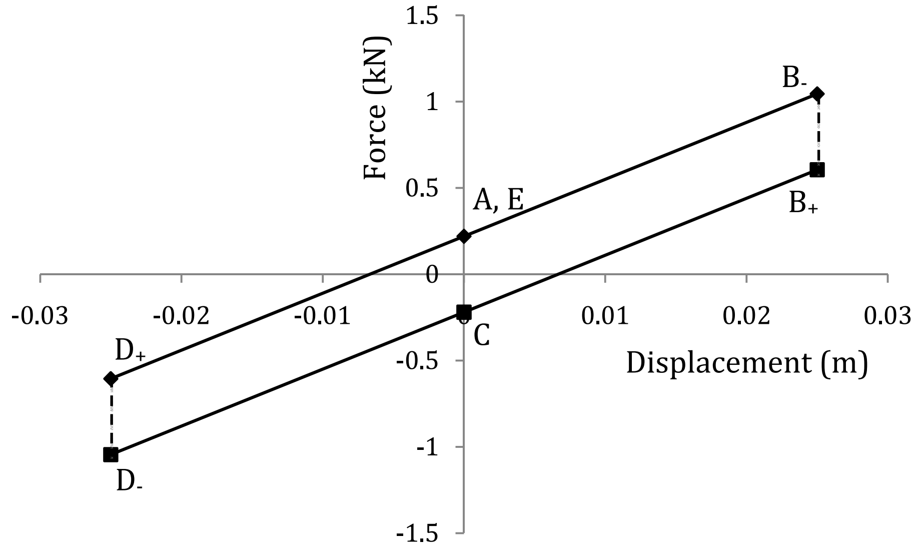


Work dissipated = $0.22(\mathrm{kN}) \times 0.025(\mathrm{m}) \times 4 = 0.022(\mathrm{kJ})$ **[2 points]**

(1.3) Draw the force - displacement diagram for the force applied to the whole shock absorber during the displacement cycle. Mark the corresponding points of A, B, C, D and E on the drawing. What is the work dissipated during this cycle?

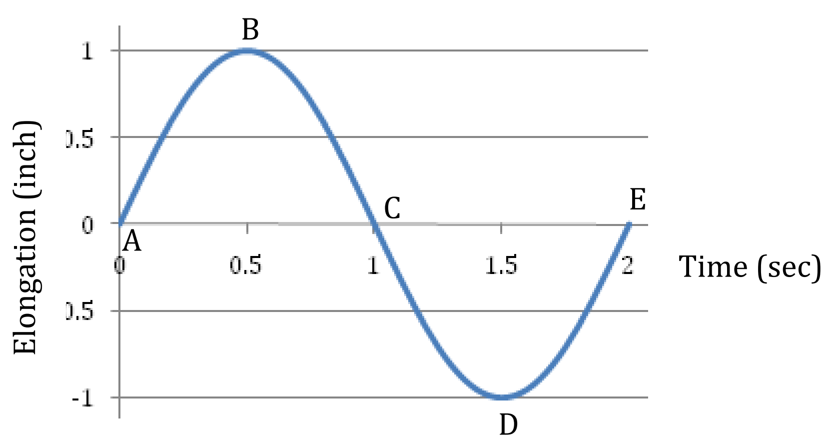
Solution:

Drawing [2 points]



Work dissipated = same as (2) $=0.022(\text{kJ})$ **[2 points]**

(2) Consider an elongation displacement cycle of the shock absorber $x(t)=C\sin \pi t$ as shown below. Here $C=1$ inch and the time t is measured in second.



(2.1) Draw the force - displacement diagram for the force applied to the spring during the displacement cycle. Mark the corresponding points of A, B, C, D and E on the drawing. What is the work dissipated during this cycle?

Solution:

Drawing [2 points]

Same as (1.1).

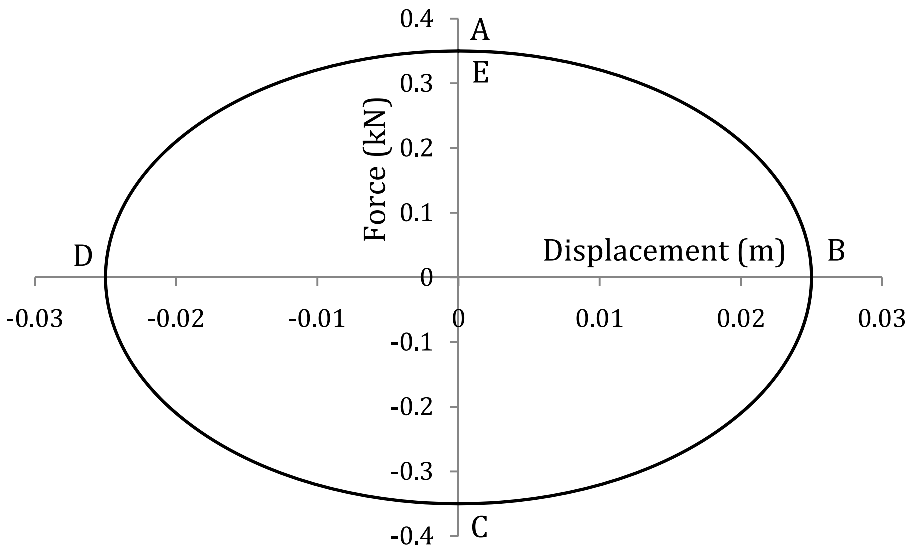
Work dissipated = same as (1.1) = 0. **[2 points]**

(2.2) Draw the force - displacement diagram for the force applied to the dashpot during the displacement cycle. Mark the corresponding points of A, B, C, D and E on the drawing. What is the work dissipated during this cycle?

Solution:

Drawing [2 points]

$$F = \lambda \times \dot{x} = 4400(\text{kg/s}) \times \pi \times 0.025(\text{m/s}) \times \cos \pi t / 1000 = 0.35 \times \cos \pi t (\text{kN})$$
$$x = 0.025 \times \sin \pi t (\text{m})$$



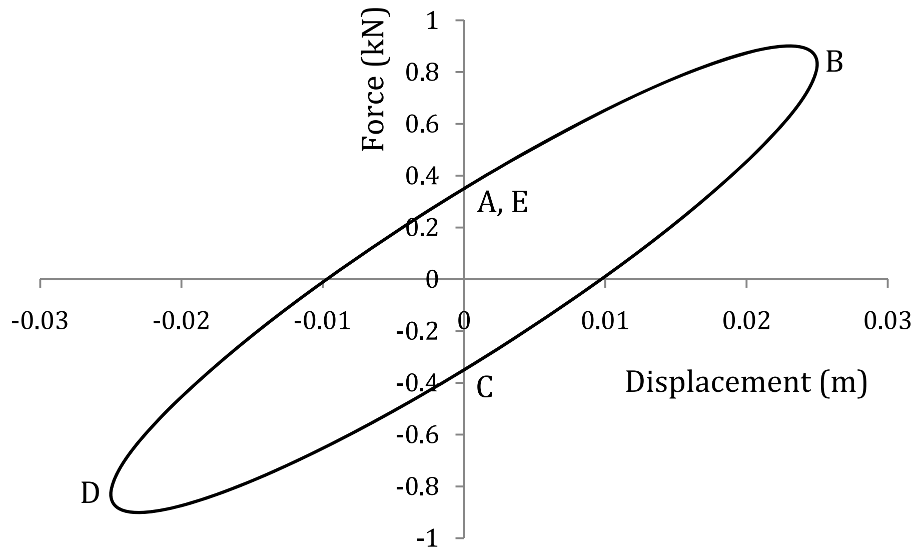
Work dissipated = $0.35(\text{kN}) \times 0.025(\text{m}) \times \pi = 0.027(\text{kJ})$ **[2 points]**

(2.3) Draw the force - displacement diagram for the force applied to the whole shock absorber during the displacement cycle. Mark the corresponding points of A, B, C, D and E on the drawing. What is the work dissipated during this cycle?

Solution:

Drawing [2 points]

$$F = kx + \lambda \dot{x} = 33(\text{kN}/m) \times 0.025 \times \sin \pi t (\text{m}) + 4400(\text{kg/s}) \times \pi \times 0.025(\text{m/s}) \times \cos \pi t / 1000$$
$$= (0.83 \times \sin \pi t + 0.35 \times \cos \pi t)(\text{kN})$$
$$x = 0.025 \times \sin \pi t (\text{m})$$



Work dissipated = $\int_{cycle} F dx = \int_0^2 (0.83\times\sin\pi t + 0.35\times\cos\pi t)\times(\pi\times 0.025\times\cos\pi t)\,dt$

$= \int_0^2 (0.35\times\pi\times 0.025\times\cos^2\pi t)\,dt\,(\mathrm{kJ}) = 0.35\times\pi\times 0.025\,(\mathrm{kJ}) = 0.027\,(\mathrm{kJ})$: Same as (2.2).

[2 points]

4. A shock absorber is driven by steady-state elongation cycles of $x(t) = C\sin\omega t$, and the corresponding steady-state net force is measured to be $F(t) = A\sin\omega t + B\cos\omega t$. Express the spring constant k and the damping coefficient λ in terms of A, B, C and ω.

Solution:

$F = kx + \lambda\dot{x} = kC\sin\omega t + \lambda C\omega\cos\omega t = A\sin\omega t + B\cos\omega t$ **[1 point]**

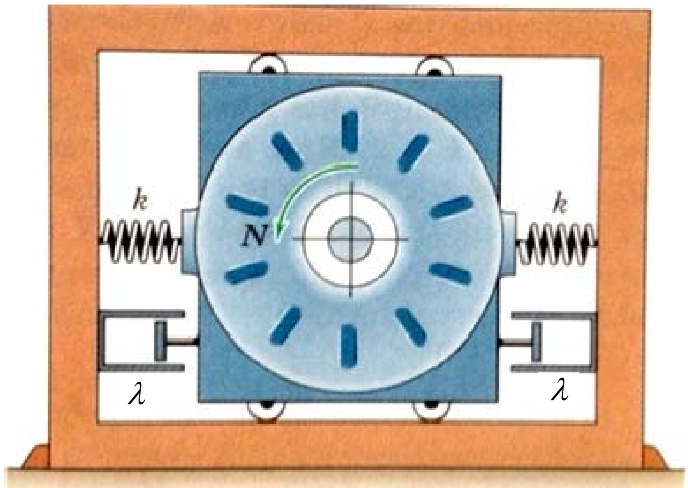
This implies $k = A/C$ **[2 points]**

and

$\lambda = B/(C\omega)$. **[2 points]**

Therefore, if you actuate a linear viscoelastic shock absorber with $x(t) = C\sin\omega t$ and measure the force $F(t) = A\sin\omega t + B\cos\omega t$, you can measure the spring constant and the damping coefficient.

5. The 20-kg variable-speed motorized unit is restrained in the horizontal direction by two springs, each of which has a stiffness of 2.1 kN/m. Each of the two dashpots has a viscous damping coefficient $\lambda = 58$ kg/s. In what range of spin speeds N (revolution/sec) can the motor run for which the magnification factor M will not exceed 2?



Solution:

The net spring constant is $2k$ **[1 point]**

and the net damping coefficient is 2λ. **[1 point]**

$$M = \frac{1}{\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[\frac{2\zeta\omega}{\omega_n}\right]^2\right\}^{1/2}} \le 2$$

$$\Rightarrow \left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[\frac{2\zeta\omega}{\omega_n}\right]^2 \ge \frac{1}{4} \quad \Rightarrow 1-2x+x^2+4\zeta^2 x \ge \frac{1}{4} \quad \text{with } x = \left(\frac{\omega}{\omega_n}\right)^2$$

or $x^2 - \left(2-4\zeta^2\right)x + \frac{3}{4} \ge 0$ **[2 points]**

$$\Rightarrow x = \left(\frac{\omega}{\omega_n}\right)^2 \le \frac{2-4\zeta^2 - \sqrt{\left(2-4\zeta^2\right)^2 - 3}}{2} \quad \Rightarrow \quad \omega \le \omega_n \sqrt{\frac{2-4\zeta^2 - \sqrt{\left(2-4\zeta^2\right)^2 - 3}}{2}}$$

or $$\left(\frac{\omega}{\omega_n}\right)^2 \geq \frac{2-4\zeta^2+\sqrt{\left(2-4\zeta^2\right)^2-3}}{2} \quad \Rightarrow \quad \omega \geq \omega_n\sqrt{\frac{2-4\zeta^2+\sqrt{\left(2-4\zeta^2\right)^2-3}}{2}}$$

Therefore, we have

$$N=\frac{\omega}{2\pi} \leq \frac{\sqrt{2k/m}}{2\pi}\sqrt{\frac{2-4\zeta^2-\sqrt{\left(2-4\zeta^2\right)^2-3}}{2}}$$ **[2 points]**

or $$N=\frac{\omega}{2\pi} \geq \frac{\sqrt{2k/m}}{2\pi}\sqrt{\frac{2-4\zeta^2+\sqrt{\left(2-4\zeta^2\right)^2-3}}{2}}$$ **[2 points]**

with $\zeta=\frac{\lambda}{\sqrt{2km}}$ for the double springs and double dashpots. **[2 points]**

Inserting the values, $\lambda = 58$ kg/s, k = 2.1 kN/m and m = 20 kg, we have

$N \leq 1.80$ revolution/sec **[1 point]**

or $N \geq 2.56$ revolution/sec. **[1 point]**