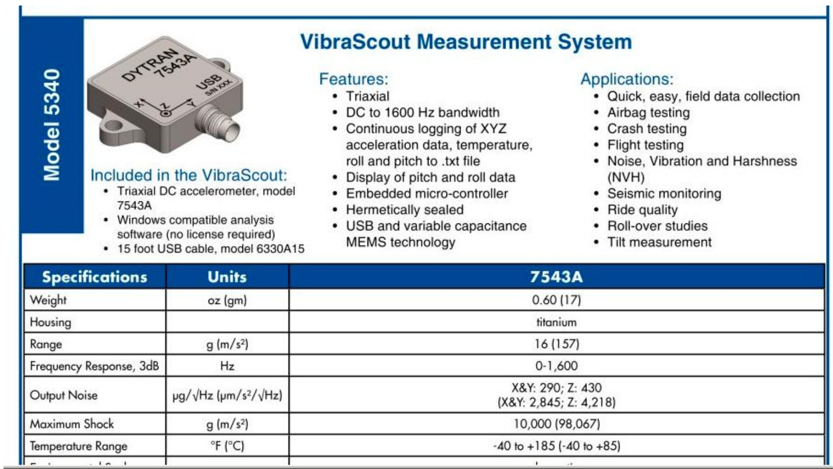
School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 2: Kinematics and Dynamics of Particles
Due Friday Feb 15, 2013
Max Score 67 Points



VibraScout Measurement System

Model 5340

Included in the VibraScout:
- Triaxial DC accelerometer, model 7543A
- Windows compatible analysis software (no license required)
- 15 foot USB cable, model 6330A15

Features:
- Triaxial
- DC to 1600 Hz bandwidth
- Continuous logging of XYZ acceleration data, temperature, roll and pitch to .txt file
- Display of pitch and roll data
- Embedded micro-controller
- Hermetically sealed
- USB and variable capacitance MEMS technology

Applications:
- Quick, easy, field data collection
- Airbag testing
- Crash testing
- Flight testing
- Noise, Vibration and Harshness (NVH)
- Seismic monitoring
- Ride quality
- Roll-over studies
- Tilt measurement

Specifications	Units	7543A
Weight	oz (gm)	0.60 (17)
Housing		titanium
Range	g (m/s²)	16 (157)
Frequency Response, 3dB	Hz	0-1,600
Output Noise	μg/√Hz (μm/s²/√Hz)	X&Y: 290; Z: 430 (X&Y: 2,845; Z: 4,218)
Maximum Shock	g (m/s²)	10,000 (98,067)
Temperature Range	°F (°C)	-40 to +185 (-40 to +85)

1. A specification sheet for a representative MEMS accelerometer (from here) is shown in the figure.

1.1 Suppose that the accelerometer measures a harmonic vibration with displacement $x(t) = X_0 \sin(\omega t)$, where X_0 is the amplitude of the vibration, and ω is its frequency. Find formulas for the velocity and acceleration as a function of time.

$$x(t) = X_0 \sin(\omega t)$$
$$\Rightarrow v(t) = \omega X_0 \cos(\omega t)$$
$$\Rightarrow a(t) = -\omega^2 X_0 \sin(\omega t)$$

[2 POINTS]

1.2 Hence, determine the maximum displacement amplitude that the accelerometer can measure at its maximum operating frequency (note that the max acceleration amplitude is given in the row labeled 'Range' in the table; the frequency range is the 'Frequency Response' row).

The max displacement is

$$X_0 = \frac{A_0}{\omega^2} = \frac{157}{(2\pi \times 1600)^2} = 1.55 \times 10^{-6} m = 1.55 \mu m$$

[2 POINTS]

2. The goal of this problem is to do a rough preliminary design calculation for Felix Baumgartner's space jump (or you could check out this reconstruction at 1/125 scale in Lego). The objectives are (i) to estimate the total duration of the jump; and (ii) to estimate his maximum speed.

We will divide the drop into two parts (i) an initial phase $0 < t < t_1$ during which Felix accelerates until he reaches his terminal velocity; and (ii) a second phase $t_1 < t < t_2$ during which he maintains his terminal velocity. Throughout this calculation y will denote Felix's height above the ground; $v = dy/dt$ and $a = dv/dt = d^2y/dt^2$ are his velocity and acceleration.

2.1 During the first phase, we will approximate his acceleration using the formula $a(t) = -g + \beta t$, where g is the gravitational acceleration, and β is a constant that we will calculate later. Find expressions for his velocity and altitude in terms of g and β, given that $y = y_0, v = 0$ at time $t=0$.

The acceleration is the time derivative of velocity. We can integrate to see that

$$\frac{dv}{dt} = -g + \beta t \Rightarrow v = -gt + \frac{1}{2}\beta t^2 \Rightarrow y = y_0 - \frac{1}{2}gt^2 + \frac{1}{6}\beta t^3$$

[2 POINTS]

2.2 When Felix is at his terminal velocity, the air drag force just balances his weight $F_D = mg$. The air drag will be approximated as

$$F_D = \frac{1}{2}\rho C_D A v^2$$

where C_D is the drag coefficient, ρ is the air density, A is Felix's projected cross-sectional area, and v is his speed. We will approximate the variation of air density with altitude by $\rho = \rho_0 \exp(-y/d)$, where ρ_0 is the air density at the earth's surface. Show that this means that during the second phase of the jump

$$\frac{dy}{dt} = -\frac{\alpha}{\exp(-y/2d)} \qquad \alpha = \sqrt{\frac{2mg}{A\rho_0 C_D}}$$

Hence find a formula for y as a function of t, d and α (you can separate variables and integrate by hand, or use Mupad). Assume that $y = y_1$ at time $t = t_1$.

Setting

$$F_D = \frac{1}{2}\rho C_D A v^2 = mg \qquad \rho = \rho_0 \exp(-y/d)$$

$$\Rightarrow \frac{dy}{dt} = v = -\sqrt{\frac{2mg}{C_D A \rho_0 \exp(-y/d)}} = -\frac{\alpha}{\exp(-y/2d)}$$

(we use the negative square root because we know the velocity is downwards). To find y we need to integrate

$$F_D = \frac{1}{2}\rho C_D A v^2 = mg \qquad \rho = \rho_0 \exp(-y/d)$$

$$\Rightarrow \int_{y_1}^{y} \exp(-y/2d)dy = -\int_{t_1}^{t} \alpha dt \Rightarrow 2d\left[\exp(-y/2d) - \exp(-y_1/2d)\right] = \alpha(t-t_1)$$

$$\Rightarrow y = -2d\log\left\{\exp(-y_1/2d) + \alpha(t-t_1)/2d\right\}$$

[3 POINTS]

2.3 To complete the calculation we need to find β, t_1 and y_1. For this purpose we will assume that at the end of the acceleration phase, $a(t_1) = 0$, and $v(t_1) = -\alpha/\exp(-y_1/2d)$. Hence show that β can be calculated from $\beta = \sqrt{g^3/6x^2 d}$, where x is the solution to the equation $x = \lambda\exp(-x^2)$ $\quad \lambda = 2\alpha\exp(y_0/2d)/\sqrt{6gd}$.

We have three unknowns and three equations. The condition $a(t_1) = 0$ gives $-g + \beta t_1 = 0$

The condition $v(t_1) = -\alpha/\exp(-y_1/2d)$ gives $-gt_1 + \frac{1}{2}\beta t_1^2 = -\alpha/\exp(-y_1/2d)$

Finally, setting $y=y_1$ in the solution to 1.1 gives $y_1 = y_0 - \frac{1}{2}gt_1^2 + \frac{1}{6}\beta t_1^3$

Substituting for t_1 from the first equation into the second, and y_1 from the first equation into the second gives

$$-g\frac{g}{\beta}t_1 + \frac{1}{2}\beta\left(\frac{g}{\beta}\right)^2 = -\alpha\exp\left\{\frac{1}{2d}\left(y_0 - \frac{1}{2}g\left(\frac{g}{\beta}\right)^2 + \frac{1}{6}\beta\left(\frac{g}{\beta}\right)^3\right)\right\}$$

$$\Rightarrow \frac{g^2}{2\beta} = \alpha\exp(y_0/2d)\exp(-g^3/6d\beta^2)$$

$$\Rightarrow \frac{1}{2}\sqrt{6gd}\sqrt{\frac{g^3}{6d\beta^2}} = \alpha\exp(y_0/2d)\exp(-g^3/6d\beta^2)$$

$$\Rightarrow x = \lambda\exp(-x^2) \qquad x = \sqrt{\frac{g^3}{6d\beta^2}} \qquad \lambda = \frac{2\alpha\exp(y_0/2d)}{\sqrt{6gd}}$$

[3 POINTS]

2.4 Assume the following values for parameters: $y_0 = 39km$, g=9.81ms^{-2}, d=5km (this value is smaller than is usually used to model the atmosphere in the troposphere – the usual estimate is d=8km - but our number gives better values in the stratosphere), $\rho_0 = 1.2$kg m^{-3}, A=1.1m^{-2}, $C_D = 0.8$, m=80kg. Use Mupad to calculate values for $\alpha, \lambda, x, \beta, t_1, y_1$. To solve the equation $x = \lambda\exp(-x^2)$ in Mupad use

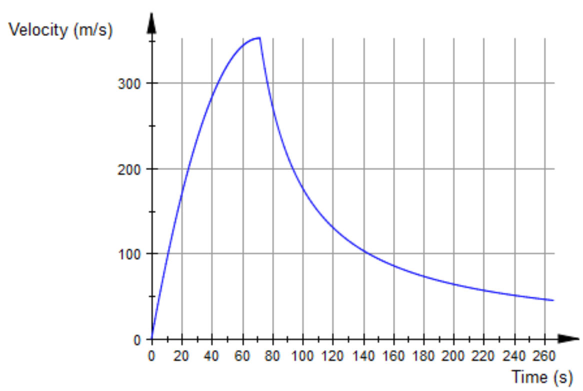
```
x := numeric::solve(xx = `&lambda;`*exp(-xx^2),xx)[1]
```

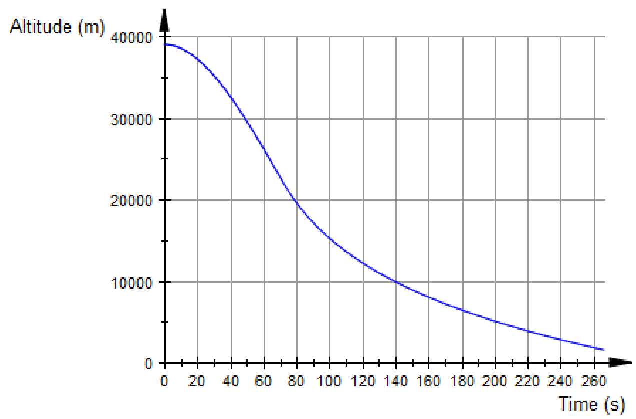
Hence, calculate the time required for Felix to drop to 5000ft (that's where the parachute was deployed), find his maximum velocity, and plot graphs showing his altitude and velocity as functions of time.

The values of all the constants (from Mupad) are

$$\alpha = 38.55 m/s$$
$$\lambda = 7.02$$
$$x = 1.299$$
$$\beta = 0.137 m/s^3$$
$$t_1 = 71.8 s$$
$$y_1 = 22125 m$$

With these parameters, we find that the jump time is 266 sec, the maximum velocity is 352 m/s. The graphs are shown below. For comparison, the actual numbers can be found on several web-sites, for example here. We get within 10% of both the measured maximum velocity and the time of the jump. There are a number of questionable assumptions in this estimate – in particular the density variation with altitude assumes that the temperature of the atmosphere is constant, (this is probably the most serious error); the acceleration in phase 1 obviously just a rough guess; the drag equation is valid only for subsonic flow, and the drag coefficient probably varies as the skydiver adjusts his profile on the way down.





Graders – note that some solutions may give negative velocities (OK!) and it's also OK to plot the graphs for 0<t<t1 and t1<t<266 separately. There may be some errors in the total time because of the conversion from ft->m – as long as the procedure is done correctly (solve the equation at the end of 2.2 for time, given *y=1520m)* the solution can get full credit. People had trouble getting the plots done – a small error in a formula or putting parentheses in the wrong place can mess everything up. Don't penalize harshly for this sort of small error – the important thing is the thought process behind the calculation)

[4 POINTS]

3. This paper describes experimental measurements of the trajectory and wing movements of a dragonfly in flight. The authors report that the position vector of the dragonfly with respect to a fixed origin can be fit by the following formulas

(a) During forward flight:

$$x = 10^{-3}(-0.3t^3 + 16.6t^2 - 52.3t - 10832.6)$$

$$y = 10^{-3}(-9.4t^2 - 1049.8t + 27657.5)$$

$$z = 10^{-3}(-1.3t^3 + 72.7t^2 - 1373.6t - 16116.6)$$

(b) During a turning maneuver

$$x = 10^{-3}(0.1t^3 - 19.3t^2 + 441.8t - 6669)$$

$$y = 10^{-3}(-1.1t^3 + 44.1t^2 - 1885.4t + 26309.8)$$

$$z = 10^{-3}(0.4t^3 - 34.2t^2 + 868.2t + 11029.6)$$

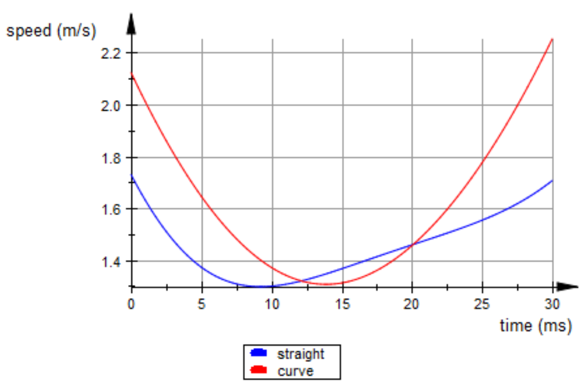
where time t is in milliseconds (during a time period $0<t<30$ms) and position is in mm. The z direction points vertically upwards. The dragonfly mass was reported as 0.29grams.

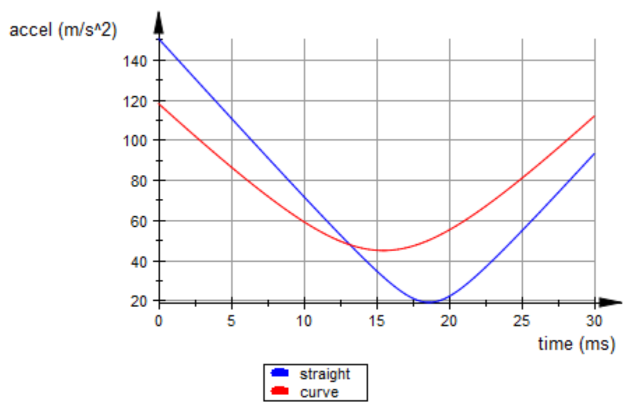
Using Mupad, plot graphs showing the following quantities, for each type of motion:

- The speed of the dragonfly (in m/s) as a function of time
- The magnitude of the acceleration (in m/s^2) of the dragonfly as a function of time. Why is the magnitude of the acceleration not zero when the slope of the speed graph is zero?
- The tangential and transverse (normal) components of the acceleration of the dragonfly. What are the times in the two experiments that best approximate straight line motion at constant acceleration, and motion along a circular path at constant speed?
- The aerodynamic force acting on the dragonfly (don't forget to include gravity).

Graders: Don't deduct points for small errors on the plots (eg if the accel units are mm/msec^2 that's OK) and if the method used to do the calculation is explained, but the plots look wrong, deduct 1/2 point for the incorrect plot but give credit for the method). Many solutions will submit the plots separately – that's fine.

The speed and acceleration magnitude plots are shown below.





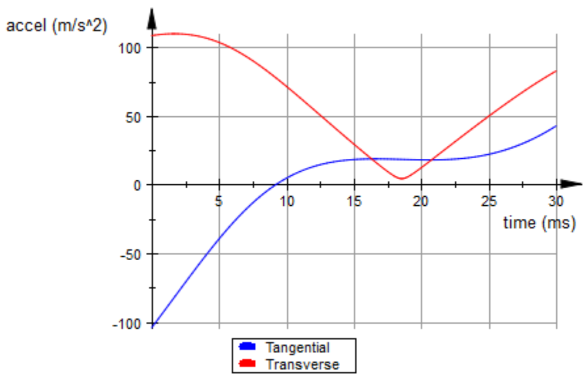
The acceleration magnitude is not zero when the slope of the speed is zero because the acceleration vector has components transverse as well as parallel to the path. The acceleration component parallel to the path quantifies the rate of change of speed, and is zero when the speed is constant (or the derivative of the speed is zero).

[3 POINTS]

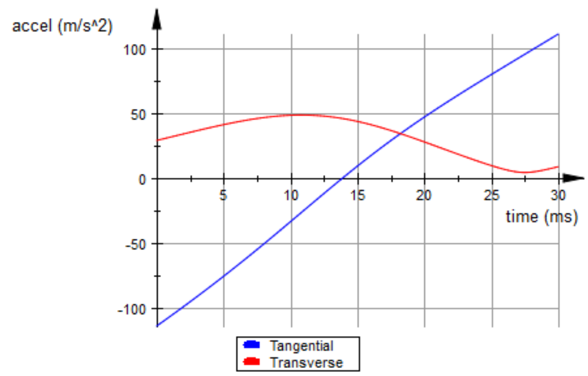
The tangential and transverse components of velocity and acceleration can be computed from the expressions

$$a_t = \mathbf{a} \cdot \mathbf{t} \qquad a_n = \left| \mathbf{a} - (\mathbf{a} \cdot \mathbf{t})\mathbf{t} \right|$$

Where $\mathbf{t} = \mathbf{v} / |\mathbf{v}|$ is a unit vector parallel to the velocity. The plots are shown below



(a) Straight

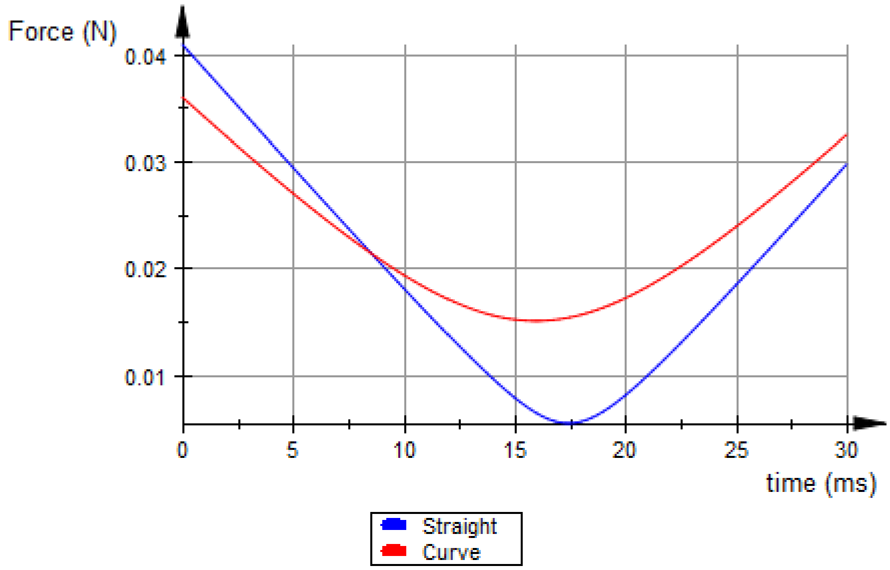


(b) Curve

In the first experiment motion is closest to straight-line motion at 17 ms (the transverse (or normal) acceleration is zero, and the tangential acceleration is approximately constant), and in the second experiment motion is closest to circular motion at constant speed at about 14ms (the tangential acceleration is zero, and the transverse acceleration is approximately constant.)
[Graders – the values for the times are not very important – if the reasoning leading to the conclusion is explained it should get full credit]

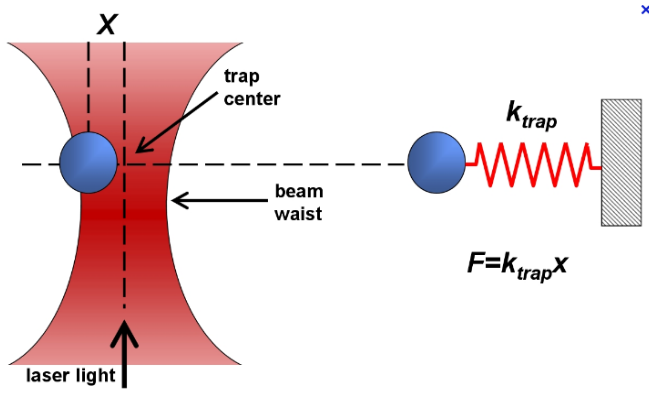
[4 POINTS]

Newton's law gives $\mathbf{F}_A - mg\mathbf{k} = m\mathbf{a} \Rightarrow \mathbf{F}_A = m\mathbf{a} + mg\mathbf{k}$, where $\mathbf{F}_A$ is the aerodynamic force. The magnitude of the aerodynamic force is plotted below



[3 POINTS]

4. Over the past two decades, people have developed a variety of techniques for manipulating very small objects.[1] One such device is an 'optical trap,' which holds small particles in place using one or more laser-beams. The underlying theory is rather complicated (a rough qualitative discussion can be found here) but essentially a particle in a light beam tends to experience a force that pushes it towards the point where the light is brightest. A particle can therefore be trapped by focusing light at a point. The particle then experiences a force $F = kr$ that is proportional to its distance r from the focal point, and acts towards the focal point.



Measuring the force exerted by an optical trap is an important problem. One way to do this is to introduce a small harmonic disturbance in the position of the focal point, so that the trapped particle moves around in a circle. If the time T for one orbit can be measured, the constant k can be determined.

4.1Show that T and k are related by

$$k = m\left(\frac{2\pi}{T}\right)^2$$

where m is the mass of the particle.

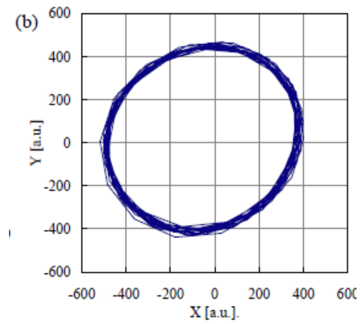
Assume that the particle moves at constant speed. Then $v = 2\pi r / T$.

[1] Many of these have yet to find commercial applications beyond basic science, so those of you with an interest in getting rich might like to try to come up with some bright ideas

For circular motion we have that $F = mv^2 / r \Rightarrow kr = \frac{m}{r}\left(\frac{2\pi r}{T}\right)^2 \Rightarrow k = m\left(\frac{2\pi}{T}\right)^2$

[3 POINTS]

4.2 As a representative example, consider the system described in Nagasaka *et al* Proc. Of SPIE Vol. 6644, who measured the orbital frequency of an 8 micrometer diameter silica sphere to be 1800 Hz (the measured orbit of the sphere is shown in the figure). Calculate the value of the constant k for the optical trap.



The density of silica is 2.66 g/cm^3 according to Google. This gives

$$m = \frac{4\pi}{3}\left(\frac{D}{2}\right)^3 \rho = \frac{4\pi}{3}\left(\frac{8\times10^{-6}}{2}\right)^3 \times \frac{2.66\times10^{-3}}{0.01^3} = 7.13\times10^{-13} kg$$

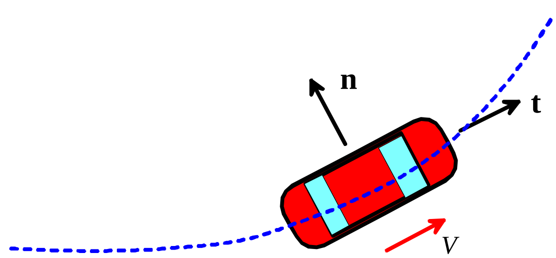
The stiffness then follows as

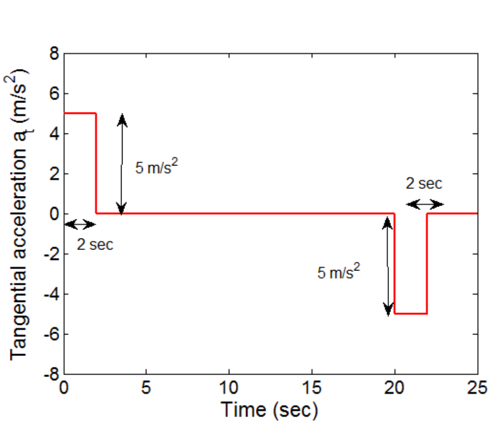
$$k = m(2\pi\times1800)^2 = 7.13\times10^{-13}(2\pi\times1800)^2 = 0.22\times10^{-3} N/m$$

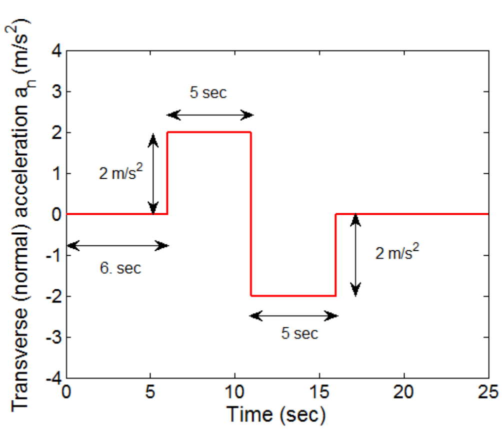
[Graders – there is likely to be some variability in values if people get densities from other sources. As long as the method is explained give full credit)

[2 POINTS]

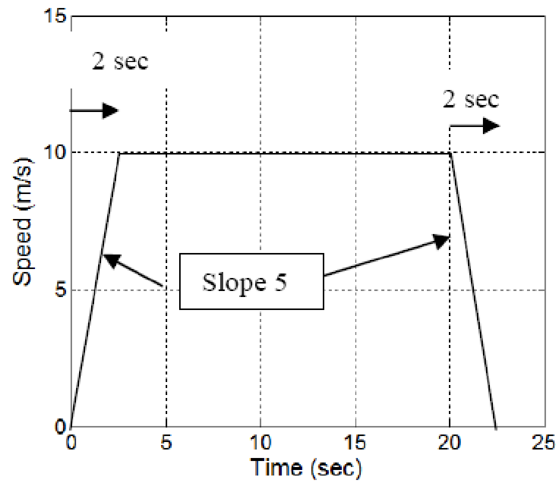
5. A car is instrumented with accelerometers that measure acceleration components a_t, a_n in directions parallel and perpendicular to the car's direction of motion, respectively. (A positive value for a_n means the car accelerates to the left). The graphs below show the variation of a_t, a_n measured in an experiment.







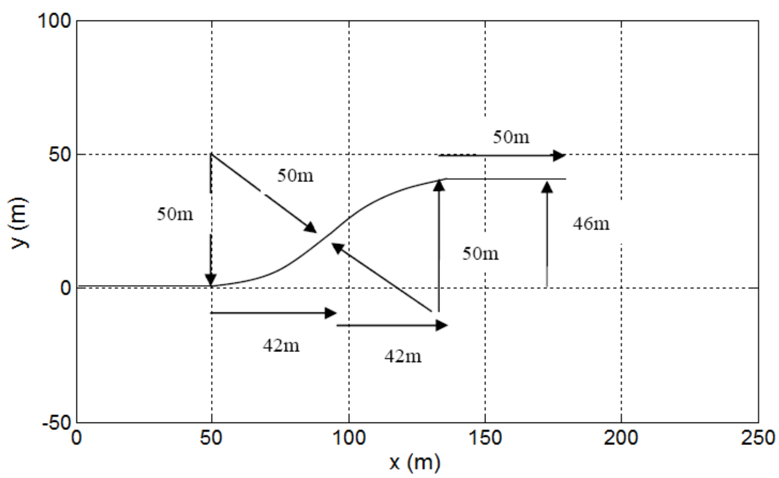
3.1 Assuming the car is at rest at time $t=0$, sketch a graph showing the car's speed as a function of time. Explain how you determined values for relevant quantities.



The speed is just the integral of the tangential acceleration – which can be determined from the area under the acceleration curve.

[2 POINTS]

3.2 Assume that at time $t=0$ the car is at the origin, and facing in the positive x direction. Sketch the subsequent path of the vehicle. Provide as much quantitative information regarding the geometry of the path as possible. Explain how you arrived at your conclusions.

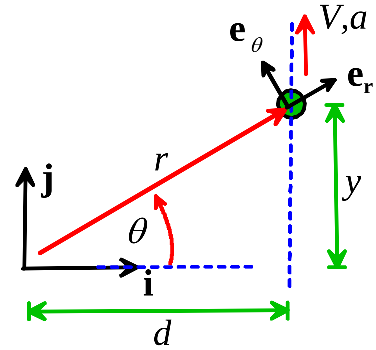


- During the first 2 sec, the car moves $10 \times 2/2 = 10m$
- During the subsequent 4 sec, the car moves $10 \times 4. = 40m$
- At $t=6s$ the car has constant speed of 10m/s and constant transverse acceleration. It must therefore move around a circular path. The constant speed circular motion formula shows that the radius of curvature of the path is $R = V^2 / a = 100/2 = 50m$. It must turn to the left, since the acceleration is positive.
- The car travels around this circular path for 5 sec – this means that it travels an arc-length 50m around the circular path. Simple geometry shows that during this segment of the path the car travels in the x direction by 50sin(1)=42m, while it travels 50(1-cos(1))=23m vertically.
- Then the path repeats in sequence but in reverse, starting with a right turn.
- The path must have a continuous slope everywhere, to ensure that the acceleration is finite (a slope discontinuity has zero radius of curvature, and if the car is moving at finite speed this gives infinite acceleration)

[Graders – solutions that calculate the radius of the curved parts and the arc-length but don't specify the 42m horizontal distance and 23m vertical distance are fine and should get full credit]

[8 POINTS]

6. This is a math-oriented problem that is meant to help you to interpret the physical significance of the various terms in the formulas for velocity and acceleration in the cylindrical-polar coordinate system. The figure shows a particle moving at instantaneous speed V and acceleration a along the line $x=d$. We could describe this motion using x,y coordinates in the $\{\mathbf{i},\mathbf{j}\}$ basis (in which case the velocity and acceleration are just $\mathbf{v}=V\mathbf{j}$ $\mathbf{a}=a\mathbf{j}$, or could use polar coordinates r,θ and the basis vectors $\{\mathbf{e}_r,\mathbf{e}_\theta\}$.



6.1 Write down formulas for the distance r and the angle θ in terms of y,d. Hence, show that

$$\frac{dr}{dt}=V\frac{y}{r}=V\sin\theta \qquad \frac{d^2r}{dt^2}=V^2\frac{d^2}{r^3}+\frac{y}{r}a=\frac{V^2}{r}\cos^2\theta+a\sin\theta$$

$$\frac{d\theta}{dt}=V\frac{d}{r^2}=\frac{V}{r}\cos\theta \qquad \frac{d^2\theta}{dt^2}=-2V^2\frac{yd}{r^4}+a\frac{d}{r^2}=\left(\frac{a}{r}-\frac{2V^2}{r^2}\sin\theta\right)\cos\theta$$

Before starting, note that $\frac{dy}{dt}=V$ $\quad \frac{d^2y}{dt^2}=a$; also that $y/r=\sin\theta$ $\quad d/r=\cos\theta$. The rest is just slogging through using all the rules of differentiation… Geometry gives

$$r=\sqrt{y^2+d^2} \qquad \theta=\tan^{-1}\frac{y}{d}$$

$$\Rightarrow \frac{dr}{dt}=\frac{y}{\sqrt{y^2+d^2}}\frac{dy}{dt}=V\frac{y}{r}=V\sin\theta$$

$$\Rightarrow \frac{d^2r}{dt^2}=\frac{1}{\sqrt{y^2+d^2}}\left(\frac{dy}{dt}\right)^2-\frac{y^2}{\left(y^2+d^2\right)^{3/2}}\left(\frac{dy}{dt}\right)^2+\frac{y}{\sqrt{y^2+d^2}}\frac{d^2y}{dt^2}$$

$$=\frac{r^2-y^2}{r^3}V^2+\frac{y}{r}a=\frac{d^2}{r^3}V^2+\frac{y}{r}a=\frac{V^2}{r}\cos^2\theta+a\sin\theta$$

$$\frac{d\theta}{dt}=\frac{1}{1+y^2/d^2}\frac{1}{d}\frac{dy}{dt}=\frac{d}{r^2}V=\frac{V}{r}\cos\theta$$

$$\frac{d^2\theta}{dt^2}=-\frac{2yd}{\left(d^2+y^2\right)^2}\left(\frac{dy}{dt}\right)^2+\frac{d}{d^2+y^2}\frac{d^2y}{dt^2}=-2V^2\frac{yd}{r^4}+\frac{d}{r^2}a=-2\frac{V^2}{r^2}\sin\theta\cos\theta+\frac{a}{r}\cos\theta$$

[4 POINTS]

6.2 Use the formulas for velocity and acceleration in cylindrical-polar coordinates to write down the velocity and acceleration vectors as components in $\{\mathbf{e}_r,\mathbf{e}_\theta\}$ in terms of v, a and θ.

For the velocity components, we have that

$$v_r = \frac{dr}{dt} = V \sin\theta$$

$$v_\theta = r\frac{d\theta}{dt} = V\cos\theta$$

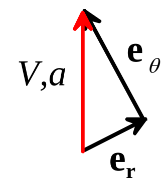
For the accelerations, we have that

$$a_r = \frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{V^2}{r}\cos^2\theta + a\sin\theta - r\left(\frac{V}{r}\cos\theta\right)^2 = a\sin\theta$$

$$a_\theta = 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = 2V\sin\theta\frac{V}{r}\cos\theta + r\left(-2\frac{V^2}{r^2}\sin\theta\cos\theta + \frac{a}{r}\cos\theta\right) = a\cos\theta$$

[2 POINTS]

6.3 Verify your answer by using the vector triangles shown in the figure below to write down the velocity and acceleration vectors as components in $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ directly.



Note that the angle between velocity/acceleration and the $\mathbf{e}_\theta$ direction is θ. Simple trig on the triangle shown gives

$$\mathbf{v} = V\mathbf{e}_r \sin\theta + V\mathbf{e}_\theta \cos\theta$$

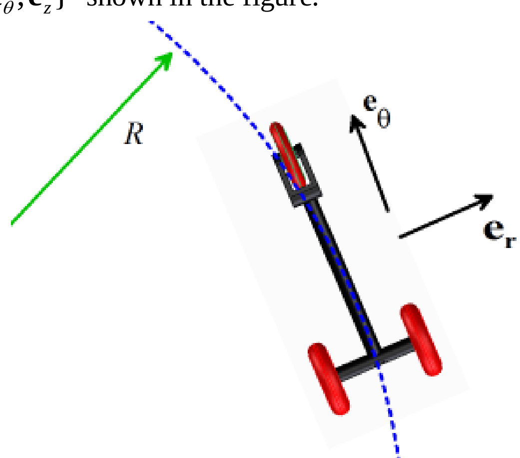
$$\mathbf{a} = a\mathbf{e}_r \sin\theta + a\mathbf{e}_\theta \cos\theta$$

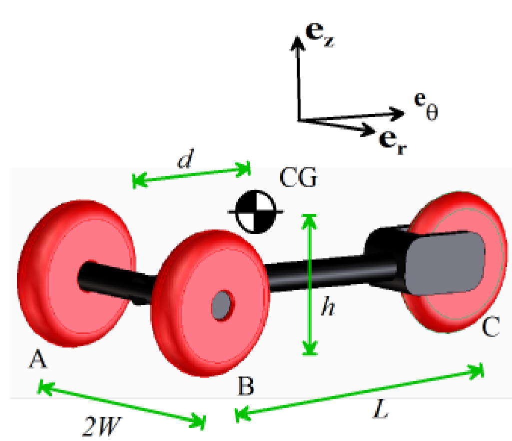
These agree with the preceding problem.

[2 POINTS]

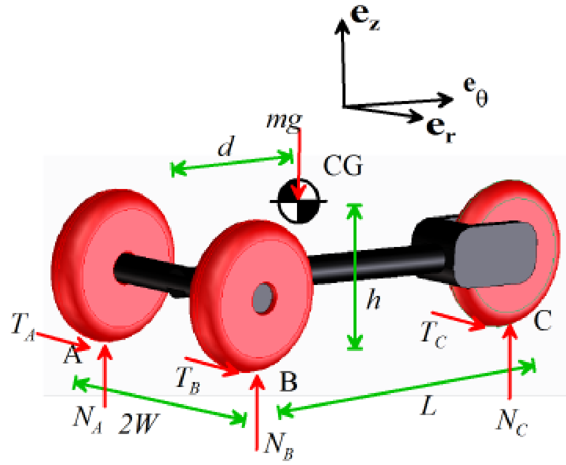
7. For safety, a tricycle should be designed so that it does not tip over in a sharp curve (tricycle riders appear to enjoy testing the design limits of their vehicles). The goal of this problem is to find the constraint relating the wheelbase L, width $2W$, and CG location of a tricycle to ensure that it will not tip (assume that the CG is a height h above the ground and a distance d ahead of the rear axle (or in many high end designs the two wheels are at the front).

Suppose that the tricycle moves around a circular path with radius R at constant speed V. Write down its acceleration as components in the cylindrical-polar basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ shown in the figure.





7.1 Draw a free body diagram showing the forces acting on the tricycle. Include friction forces at the wheels, but assume that the wheels roll without sliding.



(The friction forces T could also be drawn in the other direction – they actually act inwards, towards the center of the circular path so they actually do act the other way. But since there's no slip it doesn't matter which way we draw them, as long as they are parallel to the axle. The normal forces should be drawn upwards).

[3 POINTS]

7.2 Hence, write down Newton's law of motion for the vehicle.

$$(T_A + T_B + T_C)\mathbf{e}_r + (N_A + N_B + N_C - mg)\mathbf{e}_z = -m\frac{V^2}{R}\mathbf{e}_r$$

[2 POINTS]

7.3 Write down the position vectors of the three contact points A, B, C with respect to the center of mass. Hence, write down the equation of rotational motion.

$$\mathbf{r}_A = -W\mathbf{e}_r - d\mathbf{e}_\theta - h\mathbf{e}_z$$
$$\mathbf{r}_B = W\mathbf{e}_r - d\mathbf{e}_\theta - h\mathbf{e}_z$$
$$\mathbf{r}_C = (L-d)\mathbf{e}_\theta - h\mathbf{e}_z$$

The rotational equation of motion is $\mathbf{M}_C = \mathbf{0}$. Note that it is important to take moments about the COM.

$$\begin{aligned}\mathbf{M}_C &= (-W\mathbf{e}_r - d\mathbf{e}_\theta - h\mathbf{e}_z)\times(T_A\mathbf{e}_r + N_A\mathbf{e}_z) \\ &\quad + (W\mathbf{e}_r - d\mathbf{e}_\theta - h\mathbf{e}_z)\times(T_B\mathbf{e}_r + N_B\mathbf{e}_z) \\ &\quad + ((L-d)\mathbf{e}_\theta - h\mathbf{e}_z)\times(T_C\mathbf{e}_r + N_C\mathbf{e}_z) = \mathbf{0}\end{aligned}$$

[Graders – defining variables for the forces and positions and writing the equations using the variables is OK]

[3 POINTS]

7.4 Solve the equations of motion for the unknown horizontal and vertical reaction forces at the wheels (You can do this calculation very easily using the vector and equation solving capabilities of mupad). Note that the horizontal forces at the rear wheel cannot be determined uniquely (mupad gives the solution in terms of an arbitrary real number z.

Here’s the Mupad:

```
ra := matrix([-W,-d,-h]):
Fa := matrix([Ta,0,Na]):
rb := matrix([W,-d,-h]):
Fb := matrix([Tb,0,Nb]):
rc := matrix([0,(L-d),-h]):
Fc := matrix([Tc,0,Nc]):
Mc :=
linalg::crossProduct(ra,Fa)+linalg::crossProduct(rb,Fb)+linalg::crossProduct(rc,Fc):
eq1 := Mc=0:
eq2 := Fa+Fb+Fc+matrix([0,0,-m*g]) = matrix([-m*V^2/R,0,0]):
Forcesol := simplify(solve({eq1,eq2},{Ta,Tb,Tc,Na,Nb,Nc},IgnoreSpecialCases))
```

$$\left\{\left[Na = -\frac{m\,(\sigma_2 - L\,R\,W\,g + R\,W\,d\,g)}{2\,L\,R\,W},\ Nb = \frac{m\,(\sigma_2 + L\,R\,W\,g - R\,W\,d\,g)}{2\,L\,R\,W},\ Nc = \frac{d\,g\,m}{L},\ Ta = -\frac{L\,R\,z + L\,V^2\,m - \sigma_1}{L\,R},\ Tb = z,\ Tc = -\frac{\sigma_1}{L\,R}\right]\right\}$$

where

$$\sigma_1 = V^2\,d\,m$$

$$\sigma_2 = L\,h\,V^2$$

Simplifying slightly by hand gives

$$N_a = \frac{mg}{2}\left(1-\frac{d}{L}\right) - \frac{mV^2}{R}\frac{h}{2W} \qquad N_b = \frac{mg}{2}\left(1-\frac{d}{L}\right) + \frac{mV^2}{R}\frac{h}{2W} \qquad N_c = mg\frac{d}{L}$$

$$T_a = -\frac{mV^2}{R}\left(1-\frac{d}{L}\right) + F \qquad T_b = -F \qquad T_c = -\frac{mV^2}{R}\frac{d}{L}$$

[4 POINTS]

7.5 Assume that the contact points all have coefficient of friction μ. Find a formula for the critical speed V at which the wheels will skid.

For slip at the front wheel $|T_c| = \mu |N_c|$. The rear wheels have to both slip together, so $|T_a + T_b| = \mu |N_a + N_b|$

The rear and front wheels both skid at the same speed, regardless of the location of the COM (perhaps somewhat counter-intuitively). The critical speed is given by the formula $V = \sqrt{\mu R g}$ (see the mupad calculation listed below for the procedure to solve this problem, but its actually easier to do it by hand)

```
sliprear := -Ta-Tb = `&mu;`*(Na+Nb):
solve(subs(sliprear,Forcesol[1]),V)
```

$$\left\{R\sqrt{\frac{\mu\,g}{R}},\ -R\sqrt{\frac{\mu\,g}{R}}\right\}$$

```
slipfront := -Tc = `&mu;`*Nc:
```

$$\left\{\sqrt{\mu}\,\sqrt{R}\,\sqrt{g},\ -\sqrt{\mu}\,\sqrt{R}\,\sqrt{g}\right\}$$

```
solve(subs(slipfront,Forcesol[1]),V,IgnoreSpecialCases)
```

[3 POINTS]

7.6 Assume that the wheels do *not* skid. Find formulas for the reaction forces at the two wheels at *A* and *B*. Hence, find a formula for the critical speed required to flip the tricycle. Explain what your formula tells you about the best location for the COM of the tricycle.

The trike is at the point of tipping when the reaction force at A is zero. The critical speed for this to occur is (again, its easiest to get this by hand, but Mupad is shown below)

$$V = \sqrt{Rg}\sqrt{\frac{W}{h}\left(1-\frac{d}{L}\right)}$$

```
tip := Na=0:
solve(subs(tip,Forcesol[1]),V,IgnoreSpecialCases)
```

$$\left\{ \frac{\sqrt{R}\,\sqrt{W}\,\sqrt{g}\,\sqrt{L-d}}{\sqrt{L}\,\sqrt{h}}, -\frac{\sqrt{R}\,\sqrt{W}\,\sqrt{g}\,\sqrt{L-d}}{\sqrt{L}\,\sqrt{h}} \right\}$$

[2 POINTS]

7.7 Finally, find a formula for the maximum allowable height of the CG so that the tricycle will skid out of a turn before flipping over.

We need the skid speed to be less than the tip speed, so it occurs first.

$$\sqrt{\mu Rg} < \sqrt{Rg}\sqrt{\frac{W}{h}\left(1-\frac{d}{L}\right)} \Rightarrow \frac{W}{h}\left(1-\frac{d}{L}\right) < \mu \Rightarrow h < \frac{W}{\mu}\left(1-\frac{d}{L}\right)$$

[2 POINTS]