School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 4: Work, Energy and Linear Momentum
Due Friday March 1st

1. The 'Buckingham Potential' is used to approximate the forces acting between atoms in a diatomic molecule. The potential energy of the force of interaction between the atoms is expressed as

$$V(d) = A\exp(-Bd) - \frac{C}{d^6}$$

where *d* is the distance between the atoms. The table (from J. Bicerano "Computational Modeling of Polymers" Marcel Dekker, 1992) gives values for *A*,*B* and *C* for various bonds.

Table 3 Selected Values[a] of the Buckingham Potential Parameters *A*, *B*, and *C*.

Interaction	$10^{-3}A^b$	B^b	C^b
C···C[c]	541.4	4.59	363.0
C···C[d]	1820	4.59	556.7
N···N	393.2	4.59	547.3
O···O	135.8	4.59	217.2
S···S	906.3	3.90	3688
H···H	7.323	4.54	47.1

[a]Taken from Ref. 48.
[b]Units are such as to give energy in kcal mol^{-1} for *r* in Å.
[c]Aliphatic carbon atoms.
[d]Aromatic carbon atoms.

1.1 Plot a graph of the energy as a function of *d* for the O-O bond. Use kJoules /mol for the units of energy, and the separation *d* in Angstroms, with $2.5<d<5$. Note that 1 kcal/mol is 4.2 kJ/mol. Note also that the value of *A* in the table needs to be multiplied by 10^3

[2 POINTS]

1.2 Find an expression for the magnitude of the force acting between two atoms, and plot the force as a function of *d*, for the same range. Use eV/A (electron-volts per Angstrom – a common unit for atomic forces) for the force unit. Google will tell you the conversion factors from kJ to eV, and recall that one mol is 6.02×10^{23} molecules.

[3 POINTS]

1.3 Hence, calculate values for the following quantities for the O-O bond:

(i) The equilibrium bond length (the length of the bond when the bond force is zero)

[1 POINT]

(ii) The bond strength (the force required to break the bond)

[2 POINTS]

(iii) The binding energy (the total energy required to pull the atoms apart from their equilibrium spacing to infinity – you can express this in kJ/mol)

[1 POINT]

(iv) The stiffness of the bond (i.e. the slope of the force-separation relation at the equilibrium spacing). You can express your answer in eV/A^2

[1 POINT]

2. Japan's N700I Shinkansen 'Bullet train' has the following specifications:

- Max power output: 9760 kW
- Max weight 365 Tonnes (metric)
- Cruising speed 330 km/h

Assume that the air resistance can be approximated as

$$F_D = \frac{1}{2}\rho C_D A v^2$$

with drag coefficient $C_D = 0.1$, air density $\rho = 1.2 kgm^{-3}$ and projected frontal area $A=10\text{m}^2$

2.1 Estimate the power consumption of the train at cruise speed on level grade.

[2 POINTS]

2.2 Use the energy-power relation to calculate a formula for the acceleration of the train as a function of its speed v. Hence, calculate the shortest possible time that the train can take to accelerate to cruise speed.

[3 POINTS]

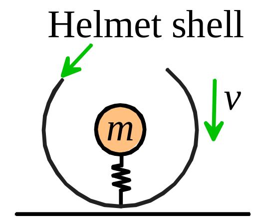
2.3 Estimate the total energy expended in overcoming air resistance in a 100km trip (you can assume constant speed). Compare the energy expended with the kinetic energy of the train at maximum speed.

[2 POINTS]

3. The standards for motorcycle and bicycle helmets are set by the Snell memorial foundation. Their latest motorcycle helmet standards can be found here. Among several other criteria, they specify that the helmets must be tested by fitting them to a head-form with mass 3.1kg, and the head-form/helmet assembly must then be dropped onto a flat anvil from a height that leads to a 7.5m/s impact velocity (see the figure for a representative experimental apparatus). The peak acceleration of the head-form during the impact must be less than 275g (where g is the gravitational acceleration)

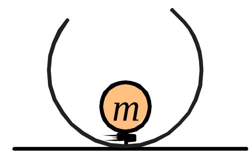
3.1 Use energy methods to calculate the required drop-height necessary to achieve the required velocity.

[2 POINTS]



3.2 The protective effect of the helmet in this impact test comes primarily from its padding. Idealize the helmet padding as a spring with stiffness k. Use energy methods to find a formula relating the maximum compression of the spring δ to the impact velocity v_0, the head form mass m and the gravitational acceleration g (assume that the outer shell of the helmet does not rebound from the anvil). Does increasing the foam stiffness k increase or decrease δ?

[3 POINTS]

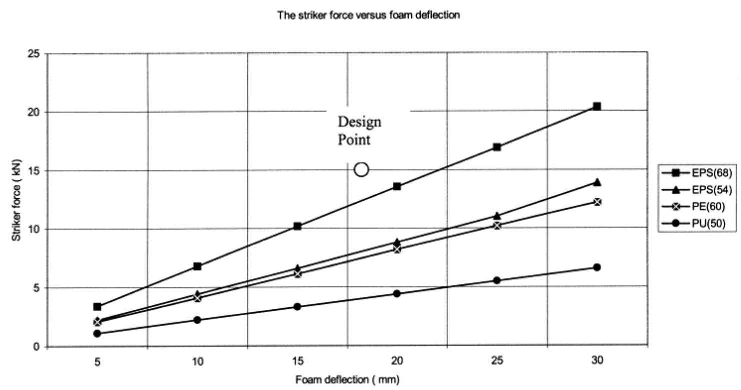


3.3 Find a formula for the maximum acceleration of the head-form in terms of the padding stiffness k, the head form mass m and the gravitational acceleration g.

[2 POINTS]

3.4 Force-deflection curves for several candidate helmet pad foams (from Shuaeib *et al*, JMEP **123,** 422, 2002) are shown in the figure (ignore the 'Design Point'). Select a foam that will minimize the padding thickness (which must exceed δ) while still meeting the Snell standard, and find the foam thickness required and the maximum predicted acceleration of the head-form.

[4 POINTS]



4. One way to measure the restitution coefficient e of an object colliding with a flat surface is to measure the total time T taken for the object to stop bouncing (of course, this only works if the object's shape permits it to bounce at all, and if the restitution coefficient is large enough for the object to bounce a fairly large number of times). In this problem you will calculate the relationship between T and e.

4.1 Suppose that the object falls from a height h above the flat surface. Use energy conservation to calculate its velocity just before impact.

[2 POINTS]

4.2 Use the restitution coefficient to determine the velocity just after impact.

[2 POINTS]

4.3 Use impulse-momentum to find the time between the first and second bounce, in terms of v_1 and g.

[2 POINTS]

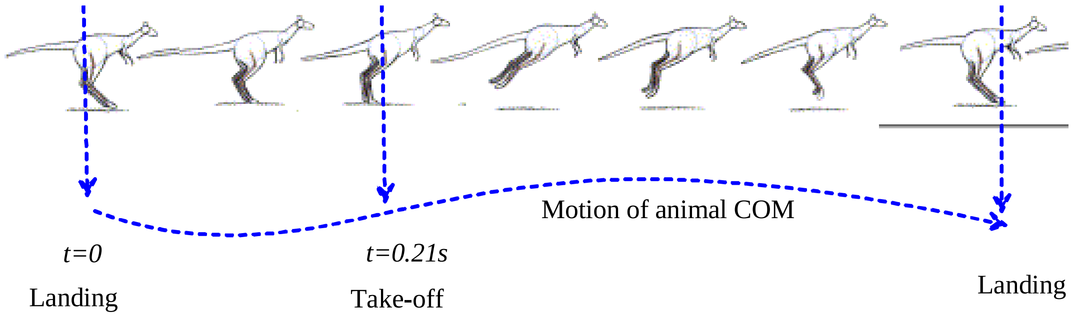
4.4 Assume that the first impact occurs at $t=0$. Show that the remaining (infinite) number of bounces occur during a total time interval

$$T = \sqrt{\frac{8h}{g}} \sum_{n=1}^{\infty} e^n$$

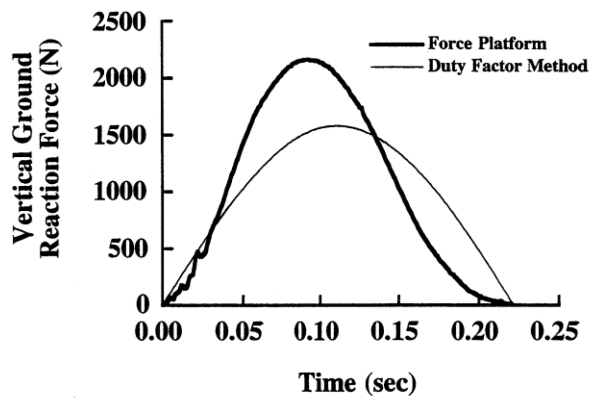
[2 POINTS]

4.5 Sum the series (mupad will do it for you) and hence find an expression for e in terms of T.

[2 POINTS]



5. The figure (from Kram *et al* Comparative Biochemistry and Physiology, **B**120 41-49, 1998) shows the variation of the contact force with time between the foot and ground of a red kangaroo as it hops. The kangaroo has a mass of 46kg and hops at a steady horizontal speed of 3.9 ms^{-1}



Suppose that the variation of contact force with time can be approximated using the equation

$$F(t) = F_0 \left(\frac{t}{t_0}\right)^2 \exp\left(-\left(\frac{t}{t_0}\right)^2\right)$$

where t_0, F_0 are two numbers that can be adjusted to give the best fit to the experimental data.

5.1 Use Mupad to plot $F(t)$, for $t_0 = 1, F_0 = 1$.

[2 POINTS]

5.2 Find a formula for the time at which $F(t)$ is a maximum, in terms of t_0, and determine the corresponding maximum force, in terms of F_0.

[2 POINTS]

5.3 Find a formula for the impulse exerted by the force, in terms of t_0, F_0

[3 POINTS]

5.4 Determine values for t_0, F_0 that will approximate the experimental data (ignore the curve labeled 'duty factor method – that's a rough theoretical prediction – see the paper for details).

[3 POINTS]

5.5 Hence, use impulse-momentum and energy conservation to estimate how high the kangaroo bounces during the jump (neglect air resistance, and assume that the kangaroo's horizontal speed is constant. Don't forget to include gravity in the impulse-momentum calculation)

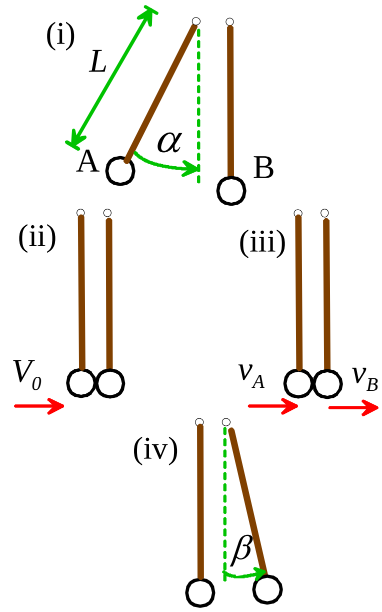
[4 POINTS]

5.6 Estimate the time that the kangaroo is airborne during one hop.

[3 POINTS]

5.7 Calculate the length of the jump (the distance traveled while airborne).

[2 POINTS]



6. The figure (from Durda *et* al, Icarus 211 849-855, 2011, see also the videos here) shows an experiment conducted by planetary geologists to determine the restitution coefficient between large masses of rock. Two large granite spheres with identical mass m are suspended from cranes to form large pendula. One pendulum is released from rest at an angle α to the vertical. It then collides with the second sphere, causing it to swing through an angle β before coming to rest. The goal of this problem is to find a formula relating α and β to the restitution coefficient.

6.1 Using energy methods, find a formula for the speed V_0 of sphere A just before impact, in terms of g and α.

[2 POINTS]

6.2 By considering the collisions, find a formula for the speed v_B of sphere B just after impact, in terms of V_0 and e.

[2 POINTS]

6.3 Find a formula for the angle β in terms of v_B. Hence, determine the required relationship between e, α, β.

[2 POINTS]

7. A recent article in Nature Physics reported observations of the collision between two 'Coronal Mass Ejections' (huge balls of magnetized plasma ejected from the sun). The authors suggest that the collision between the CMEs can be modeled using the classical theory of impact between two frictionless spheres.

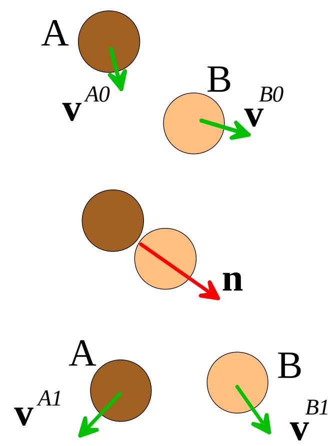


Table 1 of the article gives velocity components (in km/s) for the two colliding CMEs before (0) and after (1) collision as follows:

$$\mathbf{v}^{A0} = 237\mathbf{i} + 332\mathbf{j} \qquad \mathbf{v}^{B0} = 205\mathbf{i} + 130\mathbf{j}$$

$$\mathbf{v}^{A1} = 116\mathbf{i} + 332\mathbf{j} \qquad \mathbf{v}^{B1} = 288\mathbf{i} + 130\mathbf{j}$$

The direction of contact (**n)** is parallel to the **i** direction. The two masses are given as $m_B = 1.8 \times 10^{12} kg \qquad m_A = 1.2 \times 10^{12} kg$.

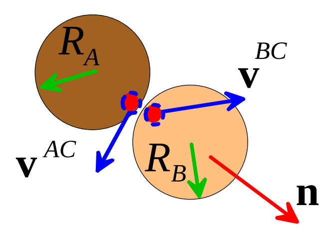
7.1Calculate the total linear momentum vector before and after the collision, and calculate the value of the restitution coefficient *e*.

[2 POINTS]

7.2 Suppose that the CMEs have the same initial velocities as those measured experimentally, but *e=1*. Calculate the predicted velocities after impact.

[3 POINTS]

7.3 The authors of the article suggest that the unusually large value for the apparent restitution coefficient is because the two CMEs are expanding at the time of collision. To check this idea, we could try to extend the 'standard' collision formulas by saying that the two points AC and BC that collide on the *surfaces* of the spheres obey



$$\frac{\left(\mathbf{v}^{BC1} - \mathbf{v}^{AC1}\right) \cdot \mathbf{n}}{\left(\mathbf{v}^{BC0} - \mathbf{v}^{AC0}\right) \cdot \mathbf{n}} = -e$$

(the argument for this is that the energy lost in a collision is determined primarily by deformation in the material in the two solids near the point of collision). We can re-write the equation for the relative velocities of the centers of the spheres as

$$\frac{\left(\mathbf{v}^{B1} - \mathbf{v}^{A1}\right) \cdot \mathbf{n} - \frac{d}{dt}(R_A + R_B)}{\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right) \cdot \mathbf{n} - \frac{d}{dt}(R_A + R_B)} = -e$$

where R_A, R_B are the radii of the spheres. The article states that the sum of the expansion speeds of the CMEs is 117 km/s. Calculate the value of the restitution coefficient *e* with this definition.

[2 POINTS]