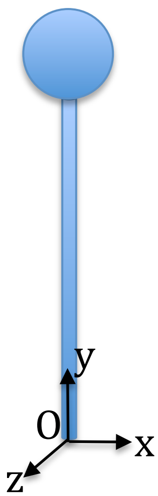
School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 8: Rigid Body Dynamics

1. CALCULATE THE MASS MOMENT OF INERTIA [6 points, 2 points per part]
You may use the mass moment of inertia tables in the slides provided online.



1A. A thin disk of radius R and mass m_d connected to a slender rod of length L and mass m_r. It rotates around the z-axis.

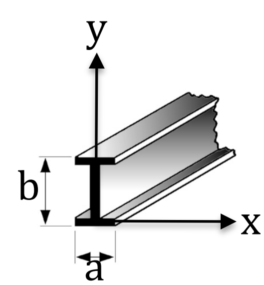
From Table:
$I_{disk} = 1/2 m_d R^2$
distance from CG $= R + L$

$I_{rod} = 1/12 m_r L^2$
distance from CG $= L/2$

$I_{total} = I_{disk} + m_d(R+L)^2 + I_{rod} + m_r L^2/4$

1B. An I-beam is composed of 3 sections of equal thickness t, density ρ, and length L. It is rotating around the y-axis.

From Table: rectangular prism $I_G = \frac{1}{12}m(a^2+c^2)$



For middle beam:
$a = t$, $c = L$, $m = \rho t L(b-2t)$
distance from CG $= L/2$
$I_G = \frac{1}{12}m(t^2+L^2) + mL^2/4 = \frac{m}{12}(t^2+4L^2)$
$I_G = \frac{\rho t L(b-2t)}{12}(t^2+4L^2)$

For top/bottom beam (both the same):
$a = a$, $c = L$, $m = \rho t a L$
distance from CG $= L/2$
$I_G = \frac{1}{12}m(a^2+L^2) + mL^2/4 = \frac{m}{12}(a^2+4L^2)$
$I_G = \frac{\rho t a L}{12}(a^2+4L^2)$

TOTAL:
$I_G = \frac{\rho t L(b-2t)}{12}(t^2+4L^2) + \frac{\rho t a L}{6}(a^2+4L^2)$

1C. A hollow cylinder of length L, outer radius R, inner radius r, and density ρ. It is rotating around the x-axis.

From Table: solid cylinder: $I_G = \frac{1}{2}MR^2$

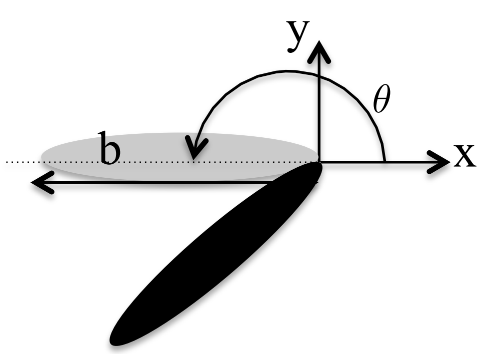
Outer Cylinder:
radius$= R$, $M_o = \rho R^2 L\rho$
$I_G = \frac{1}{2}\pi R^4 L\rho$

Inner Cylinder:
radius$= r$, $M_o = \rho r^2 L\rho$
$I_G = \frac{1}{2}\pi r^4 L\rho$

TOTAL:
$I_G = \frac{1}{2}\pi(R^4 - r^4)L\rho$

ROTATE ABOUT O:
$I_o = I_G + M_{TOTAL}R^2$
$M_{TOTAL} = \pi R^2 L\rho - \pi r^2 L\rho$
$I_o = \frac{1}{2}\pi(R^4 - r^4)L\rho + \pi(R^2 - r^2)L\rho R^2$

2. ELLIPSOID PENDULUM [10 points, 2 points per part]

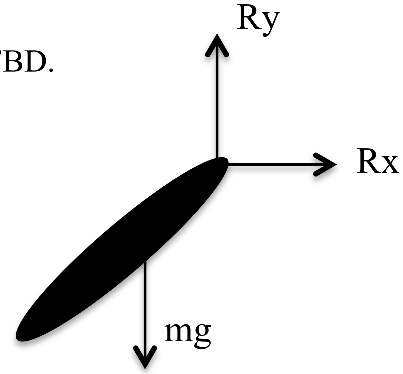


A rigid body pendulum is composed of an ellipsoid whose length is *b* and whose width and height are *a.* The ratio *b/a=10.* The ellipsoid is free to pivot about a fixed point at the origin (point O) in the *x-y* plane. It has a mass of *m*.

2A. What is the mass moment of inertia, I_o in terms of *m* and *b*?

$$I_z = \frac{1}{5}m(a^2 + b^2)$$
$$I_z = \frac{1}{5}m(\frac{b^2}{100} + b^2)$$
$$I_z = \frac{101mb^2}{500}$$
$$I_o = \frac{101mb^2}{500} + \frac{mb^2}{4}$$
$$I_o = \frac{226mb^2}{500}$$

2B. Draw a FBD.



2C. Write 3 appropriate equations of motion and identify the unknowns.

$$\sum F_x = R_x = ma_{Gx}$$
$$\sum F_y = R_y - mg = ma_{Gy}$$
$$\sum M_G = -R_y(b/2)cos\theta + R_x(b/2)sin\theta = I_G\alpha$$

OR alternatively:

$$\sum M_o = -mg(b/2)cos\theta = I_o\alpha$$

Unknowns: Rx, Ry, a_{Gx}, a_{Gy}, θ

2D. By eliminating the extra unknown variables, write a single equation of motion for the rigid body pendulum in terms of *b*, *m, g*, I_G, and θ.

$$v_o = 0$$
$$a_o = 0$$
$$a_G = a_o + (\alpha \times R_{G/o}) + (\omega \times (\omega \times R_{G/o})$$

$$a_G = -((b/2)\omega^2 cos\theta + (b/2)\alpha sin\theta)\mathbf{i} + (-(b/2)\omega^2 sin\theta + (b/2)\alpha cos\theta)\mathbf{j}$$

Substituting these expressions for a_{Gx}, a_{Gy}, and eliminating the reaction forces, we get:

$$\ddot{\theta} + \frac{mg(b/2)}{I_o} cos(\theta) = 0$$
$$\ddot{\theta} + \frac{mg(b/2)}{I_G + m(b/2)^2} cos(\theta) = 0$$

Note that there are 2 ways to express the answer: in terms of I_G or I_o. I'll accept both, as long as it's clearly noted which 'I' you are using. I intended you to work through with I_G because otherwise you don't need to use the kinematics equations because you don't need to solve for the reaction forces...

2E. For small oscillations (about the negative y-axis), what is the natural freqency in terms of *b*, *m, g*, I_G, and θ?

$$\phi = \theta - 3\pi/2 \qquad \ddot{\phi} + \frac{(b/2)mg}{I_o} sin\phi = 0$$

$$sin\phi \approx \phi \qquad \ddot{\phi} + \frac{(b/2)mg}{I_G + m(b/2)^2} sin\phi = 0$$

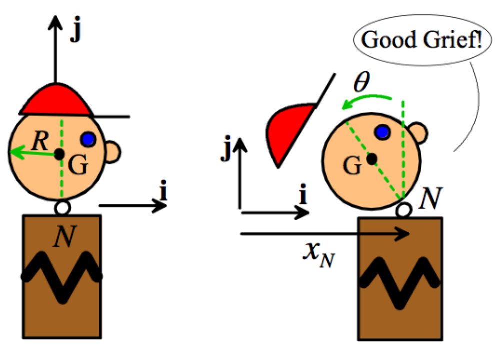
$$\omega_n = ((b/2)mg/(I_o))^{1/2} = ((b/2)mg/(I_G + m(b/2)^2))^{1/2}$$

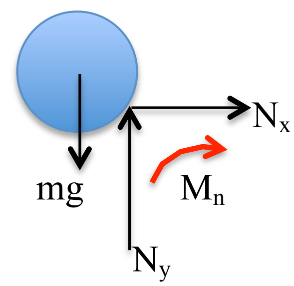
3. PASSENGER WITH NO HEADREST [6 points, 2 points / 4 points]

The figure shows a passenger in a vehicle with no headrests. The car is initially at rest, and is hit from behind by another vehicle, giving the car and the passenger's torso a forward acceleration of $\mathbf{a} = a_N\mathbf{i}$. The acceleration bends the passenger's neck through an angle θ as shown in the figure.

- Assume the head is a sphere with radius R, mass m, and mass moment of inertia of $I_G=2mR^2/5$.
- There is a moment applied by neck on the head. Assume the next acts as a torsional spring with spring constant *k*.

3.1 Draw a FBD of the head.





3.2 Show that the equation of motion is

EOM:

$$N_y - mg = ma_y$$

$$N_x = ma_x$$

$$M_n + N_x R cos\theta + NyR sin\theta = \frac{2}{5} mR^2 \alpha$$

$$M_n = -k\theta$$

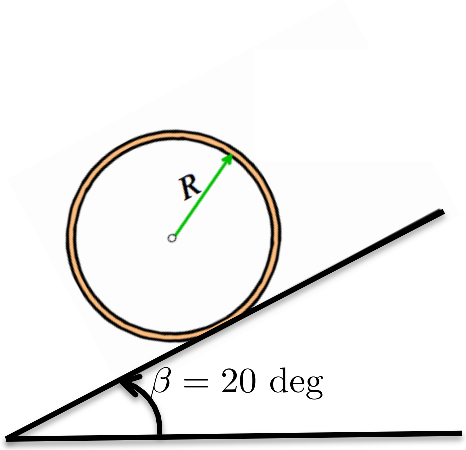
KINEMATICS:

$$a_x = -R\omega^2 sin\theta + a_N - R\alpha cos\theta$$

$$a_y = R\alpha sin\theta - R\omega^2 cos\theta$$

SOLVE (see MUPAD attached)

4. ROLLING RING [12 points, 2 points each]



Consider a ring of radius mass M and radius R, initially released from rest on an incline of 20 degrees. The coefficient of kinetic friction is μ_k, and the coefficient of static friction is μ_s.

4.1 Assuming a thickness *t* that is much smaller than *R*, derive an expression for the mass moment of inertia in terms of *M* and *R*.

assume all the mass is located a distance R from the CG, thus directly applying the definition, we have:

$I_G = MR^2$

4.2 Let's first assume the ring rolls without slipping. Derive an expression for the linear acceleration of the center, plus the angular acceleration soon after it begins to roll.

$$mgsin\beta - f = ma_x$$
$$N - mgcos\beta = 0$$
$$fR = MR^2\alpha$$

4 unknowns: f, N, a_x*,* α. Need 1 kinematic relationship: $a_x = -R\alpha$.

$$a_x = (g/2)sin\beta$$
$$\alpha = -(g/2R)sin\beta$$

4.3 If R=0.5m, μ_k=0.12, and μ_s=0.15, does our assumption of rolling without slip hold? Why or why not?

Friction force: $f = \mu_s N = \mu_s mg\ cos\beta = 0.141\ mg$ = This is the maximum possible friction force for the ring to spin without slipping!

Now, from the x-momentum equation: $f = mg\ sin\beta - ma_x = mg\ sin\beta - m(g/2)\ sin\beta = 0.171\ mg$*!* THIS IS BIGGER THAN THE MAX POSSIBLE STATIC FRICTION! Thus, the ring must slip while it rolls!

4.4 For the given values in 4.3, how long will it take the ring to travel 5 meters down the incline?

Now we have to solve the slipping/rolling problem, we now have kinematic friction: $f = \mu_k N$ *(in the direction of uphill – same as before)* x-momentum equation yields: $mg\ sin\beta - \mu_k mg\ cos\beta = ma_x$

$a_x = g\ (sin\beta - \mu_k\ cos\beta) = 2.25\ m/s^2$

$x = a_x\ t^2/2$

$t = (2*5/\ a_x)^{1/2} = 2.1\ sec$

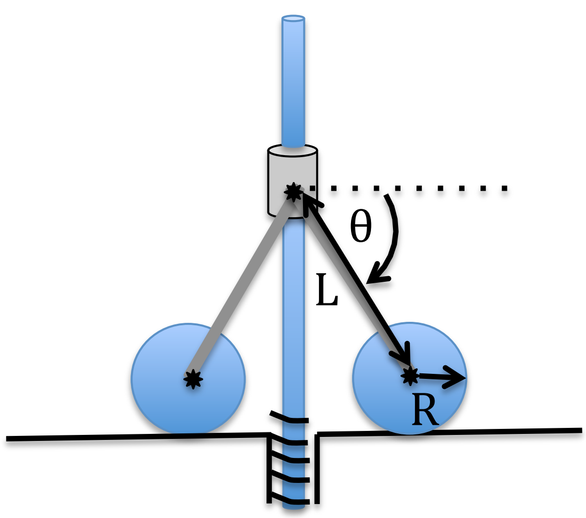
4.5 Now let's assume it is *rolling without slip*. Using energy methods, calculate the rotational velocity after it has travelled 5 meters.

$$0.5mV^2 + 0.5I_G\omega^2 = mg5sin\beta$$
$$0.5mV^2 + 0.5mR^2V^2/R^2 = mg5sin\beta$$
$$V^2 = g5sin\beta$$
$$V = (g5sin\beta)^{1/2}$$
$$\omega = (g5sin\beta)^{1/2}/R$$
$$\omega = 8.2rad/s$$

4.6 In problem 4.5, is there work done by friction? Why/Why Not?

No – there is no velocity at the contact point, thus no work done by friction!

5. Spring Collar Wheel System [6 points, 3 points each]



In the mechanism shown, each of the two wheels (mass=m_w, radius=R) is connected to a collar via a slender bar (length=L, mass=m_b). The collar slides (mass=m_c) frictionlessly on a vertical shaft, and hits a spring (spring constant=k) when the bars are in a horizontal configuration. The wheels roll without slipping.

If the collar is released from rest at θ=45 deg:

5.1 Calculate the velocity of the collar as it first hits the spring.

Wheels: $T_i=0,\ T_f=0$ (momentarily zero while the wheels change direction)
Collar: $T_i=0,\ T_f=0.5\ m_c\ v_c^2,\ V_i= m_cgH,\ \ H = L/sqrt(2),\ V_f= 0$

Bars: $T_i=0$, $T_f=2(0.5\ I_o\ \omega^2)= I_o\ \omega^2$ (this is pure rotation about the fixed axis at the center of the wheel "O" which is true when θ=0)
$I_o=1/3\ m_bL^2$
$\omega^2=v_c/L$
$T_f=1/3\ m_b\ v_c^2$
$V_i= 2m_bgH/2= m_bgH$
$V_f= 0$

Adding all 4 components:
$m_cgH + m_bgH =1/2\ m_c\ v_c^2 + 1/3\ m_b\ v_c^2$

$v_c = gH(m_c+m_b) / (m_c/2 + m_b/3)$

5.2. Calculate the maximum deformation of the spring.

At the instant the spring is at its maximum deflection, there is no kinetic energy (nothing is moving). Thus, we have gravitational potential energy that is converted to spring energy:

$V_i =m_c g\ (H + x) + 2m_b g(H/2 + x/2)$
$V_f =1/2\ kx^2$

$x = (2\ [\ m_c g\ (H + x) +2m_b g(H/2 + x/2)]\ /\ k\ )^{1/2}$ where $H = L/sqrt(2)$